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Kapitel: §3. Z/4-MANIFOLDS

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$$\begin{array}{ccccccc} & & H^2(\mathbf{Z}/p^2;\Lambda_1) & \to & H^2(\mathbf{Z}/p^2;M) & \to & 0 \\ & & \downarrow {\scriptstyle \Delta = i*(\Lambda_1)} & & \downarrow {\scriptstyle i*(M)} & & \\ H^2(\mathbf{Z}/p;\varphi_2\Lambda_2) & \overset{\lambda^k_*}{\to} & H^2(\mathbf{Z}/p;\Lambda_1) & \to & H^2(\mathbf{Z}/p;M) & \to & 0 \end{array}$$

where Δ is the diagonal map $\mathbf{Z}_p \to p(\mathbf{Z}/p)$. Hence to eliminate $M_6(k)$, it suffices to show $\mathrm{Im}(\Delta) \subset \mathrm{Im}(\lambda^k_*)$. Let e denote a column p-vector consisting of all 1's, according to the proof of (2.5) we must find an $\bar{x}_k$, $1 \leqslant k \leqslant p-1$ so that $C_{p,k} \cdot \bar{x}_k = C^k_{p,1} \cdot \bar{x}_k = e$. We do this inductively on k. For example, $\bar{x}_1 = \begin{bmatrix} 1 \\ 2 \\ \cdot \\ \cdot \\ \cdot \\ p \end{bmatrix}$, as can easily be checked. Inductively we define

$$\bar{x}_k(i) = \begin{cases} 1 & i = 1 \\ \bar{x}_k(i-1) + \bar{x}_{k-1}(i) & i > 1\,. \end{cases}$$

Clearly $C_{p,1} \cdot \bar{x}_k = \bar{x}_{k-1}$, for all coordinates except possibly the first; we must show $\bar{x}_k(p) \equiv 0 \pmod p$. But a comparison of the $\bar{x}_k$'s with Pascal's triangle convinces one that

$$\bar{x}_k(p) = \binom{p-1+k}{p-1} \equiv \binom{k-1}{p-1}\binom{1}{0} \equiv 0 \pmod p\,,$$

since $k-1 < p-1$.

We leave it for the reader to check that the restriction maps for $M_7(k)$ and $M_9(k)$ are non-trivial.

§ 3. Z/4-MANIFOLDS

In this section, we consider the case $p = 2$. For convenience, we change the notation slightly and write M_7 for $M_6(1)$ and M_i for $M_i(0)$, $i = 6, 8, 9$. According to (2.7), the indecomposable $\mathbf{Z}/4$-lattices that carry special classes are M_1, M_4 and M_9. It is easy to see M_i is faithful if and only if

$i = 3, 5, 6, 7, 8, 9$. Hence if $M = \sum m_i M_i$ is an arbitrary $\mathbf{Z}/4$-lattice then M is a faithful representation carrying a special class if and only if the multiplicities m_i satisfy the inequalities:

$$m_1 + m_4 + m_9 > 0 \tag{3.0}$$
$$m_3 + m_5 + m_6 + m_7 + m_8 + m_9 > 0\,.$$

Since the multiplicities are a complete set of isomorphism invariants in the case $p = 2$ (see section 1) one can use the conditions (3.0) to show:

(3.1) THEOREM. *If $L_n(m)$ denotes the number of isomorphism classes of n-dimensional $\mathbf{Z}/m$-lattices that carry special classes, then:*

$$L_n(4) = \sum_{j=2}^{n-1}\left(a_j - \left[\frac{j}{2}\right] - 1\right) + \sum_{j=[n]_2+2}^{n-2}(a_j - a_{j-1} - 1) + \sum_{j=[n]_4}^{n-4}(a_j - a_{j-2} - a_{j-4} + a_{j-6})$$

where $[k]_p$ denotes the reduction of k modulo p, $[k]$ denote the largest integer $\leqslant k$ and the a_j's are given by

$$P(t) = \sum_{j=0}^{\infty} a_j t^j = \frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^3}$$

In particular, the number of n-dimensional $\mathbf{Z}/4$-manifolds is at least $L_n(4)$.

Proof. If $Q(t)$ is a power series, let $\operatorname{coef}(n, Q(t))$ denote the coefficient of t^n in $Q(t)$. The number $L_n(4)$ counts the number of ways of writing

$$n = m_1 + m_2 + 2(m_3 + m_4) + 3(m_5 + m_8) + 4(m_6 + m_7 + m_9)$$

where the m_i's satisfy (3.0). If $m_1 > 0$ there is a contribution:

$$\sum_{m_1=1}^{n-2} \operatorname{coef}(n-m_1, P(t)) - \left(\left[\frac{n-m_1}{2}\right] + 1\right)$$

where $\left[\frac{n-m_1}{2}\right] + 1$ is the number of ways of expressing $n - m_1$ as a combination of 1's (M_2) and 2's (M_4) (not permitted by (3.0)). Reindexing gives the first term for $L_n(4)$.

Similarly, if $m_1 = 0$, $m_4 > 0$ there is a contribution:

$$\sum_{m_4} \operatorname{coef}(t^{n-2m_4}(1-t)P(t)) - 1$$

where 1 is subtracted to omit choosing m_2 alone. Finally, if $m_1 = m_4 = 0$, we have:

$$\sum_{m_9} \operatorname{coef}(t^{n-4m_9}, (1-t^2)(1-t^4)P(t)).$$

The coefficients of $(1-t)P(t)$ and $(1-t^2-t^4-t^6)P(t)$ are easily expressible in terms of the a_j's and the result follows.

Remark. In order for a $\mathbf{Z}/p$-lattice to carry a special class, the multiplicity of the trivial representation must be non-zero. Topologically this is reflected in the fact that a $\mathbf{Z}/p$-manifold fibers over a circle. This is already false for a 4-dimensional $\mathbf{Z}/4$-manifold as the following example shows.

Example. $L_4(4) = 6$. The multiplicities of the indecomposables in these 4-dimensional $\mathbf{Z}/4$-lattices are given by:

notation of [2]	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
07/02/02	1				1				
12/01/04	1							1	
12/01/02	1	1	1						
07/02/01	2		1						
12/01/03			1	1					
12/01/06									1

where the first column gives the label of these "**Z**-classes" from the table of the four-dimensional crystallographic groups in [2]. In fact, as these tables indicate, there is precisely one $\mathbf{Z}/4$-manifold corresponding to each $\mathbf{Z}/4$-lattice, hence there are exactly 6 4-dimensional $\mathbf{Z}/4$-manifolds.

Remark. Recall that if $p < 23$, the field $\mathbf{Q}(e^{2\pi i/p})$ has class number one. This fact, along with the work of Charlap [4], shows that the number of n-dimensional $\mathbf{Z}/p$-manifolds is exactly $L_n(p)$, $p < 23$. This number is readily computable, as Charlap [4, p. 30] remarks, and the precise formula is:

$$L_n(p) = \sum_{j=p-1}^{n-1} \left(\left[\frac{j}{p-1} \right] - \left< \frac{j}{p} \right> + 1 \right) \tag{3.2}$$

where $\langle k \rangle$ denotes the smallest integer $\geqslant k$. In particular, $L_p(p) = 1$, $L_n(p) = 0$, $p > n$, and when $p = 2$

(3.3) $$L_n(2) = \left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right) - 1 + \frac{[n]_2}{4}$$

using the notation of (3.1).

One can easily construct the following table of values of $L_n(p)$:

n	2	3	4	5	6
p					
2	1	3	5	8	11
3		1	2	3	4
5				1	2

Hence 14 of the 74 4-dimensional flat manifolds have cyclic holonomy $\leqslant 5$. (Furthermore, 26 have holonomy the Klein 4-group.) We describe analogous facts in dimension 5 below.

We let $SH^2(H, M)$ denote the set of special classes in $H^2(H, M)$. If H is a cyclic p-group and $i: \mathbf{Z}_p \hookrightarrow H$ is the inclusion of the subgroup of order p, then

$$SH^2(H, M) = H^2(H, M) - \ker(i*).$$

If N (resp. Z) denotes the normalizer (resp. the centralizer) of H in $\mathrm{Aut}(M)$, there is an exact sequence (see [15, p. 50])

$$0 \to Z \to N \to \mathrm{Aut}(H).$$

We conjecture:

Conjecture. If $\mathbf{Z}[e^{2\pi i/p^k}]$ is a unique factorization domain for, $1 \leqslant k \leqslant n$, then N acts transitively on $SH^2(\mathbf{Z}/p^n; M)$ for any H-lattice M.

The case $n = 1$ of the Conjecture follows from Charlap [4]. Class number tables shows that the $n = 2$ case applies to $p = 2, 3, 5$, the $n = 3$ case to $2, 3$ and the $n = 4$ case to $p = 2$. This conjecture implies that the lower bound of (3.1) is exact.

We mention that the multiplicities of the indecomposables in the 5-dimensional $\mathbf{Z}/4$-lattices that admit special classes are given by:

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9
	1								1
	1						1		
*						1			
	1	1			1				
*	1	1						1	
	1		1	1					
*	1		2						
*	1	2	1						
*	3							1	
	2				1				
*	2	1	1						
	3		1						
*		1	1	1					
				1	1				
*				1				1	
*		1							1

Those lattices that are starred clearly satisfy the Conjecture.

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