HODGE DECOMPOSITION ON STRATIFIED LIE GROUPS

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1. INTRODUCTION AND HISTORY

The Hodge decomposition theorem is the following:

THEOREM. *On a compact Riemannian manifold every p-form,* α, *can be written as* $\alpha = \alpha_1 + \alpha_2 + \alpha_3$ *where* $\alpha_1 = d^*\beta_1$, $\alpha_2 = d\beta_2$ *and* α_3 *is harmonic.*

This result appears in Hodge's book *The Theory and Applications of Harmonic Integrals* (1941) [12]. Since the appearance of this result generalizations of the theorem have been proven in new settings. Kodaira (1949) extended the result to certain forms on non-compact Riemannian manifolds [13] and Dolbeault (1953) derived a similar decomposition for Hermitian manifolds [5]. Atiyah and Bott (1967) defined an elliptic complex which generalized the de Rham and Dolbeault complexes [1]. In a different vein Spencer outlined a program to solve overdetermined equations (1963) [17]. The heart of his program was to obtain a Hodge decomposition paying special attention to boundary values.

Boundary value problems in complex analysis led to the $\bar{\partial}_b$ complex. It was first studied by H. Lewy (1957) [15] and generalized by Kohn and Rossi (1965) [14] and by Greenfield (1968) [10]. The complex is not elliptic but it does enjoy certain properties of elliptic complexes. For instance, its Laplacian, $\Box_b$, (with respect to a Hermitian metric) is hypoelliptic, i.e., if $\Box_b f = g$ and g is C^∞ on an open set U, then f is C^∞ in U. Folland and Stein (1973, 1974) [7, 8] wrote down an explicit fundamental solution for $\Box_b$ on the Heisenberg group. The group is not compact so Kodaira's arguments to obtain the decomposition do not apply. One of the aims of this paper is to exploit the simple homogeneity properties to obtain a fundamental solution. The technique generalizes to a class of nilpotent groups called stratified groups introduced by Folland (1975) [9]. (Also see Rothschild and Stein [16].)

The Hodge decomposition for the $\bar{\partial}_b$ complex on the Heisenberg group appears in [11] by Harvey and Polking and in [4]. The second reference motivates the technique used here. Harvey and Polking use complex analysis to obtain their result (solving the $\bar{\partial}_b$ problem first, then the $\Box_b$ problem). Using their techniques Dadok and Harvey [2] have found a fundamental solution for $\Box_b$ on the sphere in $\mathbf{C}^n$. A parametrix for $\Box_b$ on the sphere also appeared in [4] but will not be presented here, due to the more complete result of Dadok and Harvey.

Let us briefly review the Hodge decomposition. For the classical version see [3] and [12]. Let M be an n-dimensional C^∞ manifold and let E and E' be vector bundles over M whose fibers are isomorphic to $\mathbf{F}^m$ and $\mathbf{F}^{m'}$, respectively. (We let $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$.) We denote the space of smooth sections of E by $C^\infty(M, E)$ and when there is no confusion we abbreviate the notation to $C^\infty(E)$. A differential operator is a map $D: C^\infty(E) \to C^\infty(E')$ such that given any local trivializations of E and E' over U (where $U \subset M$ is open), D can be expressed by an $m' \times m$ matrix of differential operators defined on $\mathbf{F}$-valued functions on $\mathbf{R}^n$. See [18] for details.

Suppose we are given three vector bundles, E_1, E_2, and E_3 over M and differential operators $D_1: C^\infty(E_1) \to C^\infty(E_2)$ and $D_2: C^\infty(E_2) \to C^\infty(E_3)$. If $D_2 \circ D_1 = 0$ we say that the complex

$$C^\infty(E_1) \xrightarrow{D_1} C^\infty(E_2) \xrightarrow{D_2} C^\infty(E_3) \tag{1}$$

is a differential complex. Examples of differential complexes are the de Rham, Dolbeault, and $\bar{\partial}_b$ complex.

Assume there exists a measure $d\mu$ on M and a metric on the E_i which we denote by $(\cdot, \cdot)_{i,x}$ where $x \in M$. For $f, g \in C^\infty(E_i)$, one of which is compactly supported, define

$$(f, g)_i = \int_M (f(x), g(x))_{i,x} d\mu(x) .$$

Define the formal adjoint, D_1^*, of D_1 by the identity

$$(f, D_1 g)_2 = (D_1^* f, g)_1$$

where $f \in C^\infty(E_2)$ and $g \in C_c^\infty(E_1)$. Note that $C_c^\infty(E_i)$ is the subset of compactly supported elements of $C^\infty(E_i)$. Similarly, we define D_2^*. The Laplacian is given by

$$\Delta = D_1 D_1^* + D_2^* D_2 .$$

Let H be the kernel of Δ in $C_c^\infty(E_2)$. A Hodge decomposition for $C_c^\infty(E_2)$ is

$$C_c^\infty(E_2) = D_1(C_c^\infty(E_1)) \oplus D_2^*(C_c^\infty(E_3)) \oplus H .$$

Hodge studied the de Rham complex on a compact Riemannian manifold. The Riemannian metric induced the metrics on the bundles $\Lambda^p T^*(M)$ as well as the volume element.

In the next section we discuss abstract CR manifolds and look at the Heisenberg group in detail. We write down the $\bar\partial_b$ and $\Box_b$ operators explicitly and give Folland and Stein's inverse to $\Box_b$. In section 3 we introduce the stratified Lie groups and the associated homogeneous structures. We present the continuity theorems of Folland and Rothschild and Stein for convolution operators. In section 4 we prove the decomposition theorem in the general setting of stratified groups.

These results are an extension of the author's dissertation [4]. We wish to express our deep gratitude to M. Kuranishi. We would also like to thank D. Tartakoff for his help and encouragement.

2. CR STRUCTURES AND THE HEISENBERG GROUP

Let M be a C^∞ manifold of dimension $2n + 1$. The complexified tangent bundle of M, $\mathbf{C}T(M)$, is the bundle whose fiber is $\mathbf{C} \otimes_{\mathbf{R}} T_m(M)$ where $T_m(M)$ is the tangent space at $m \in M$. When there is no confusion we will drop reference to M in the notation for $T(M)$, $\mathbf{C}T(M)$, etc. So, $T(M) = T$ and $\mathbf{C}T(M) = \mathbf{C}T$, for example.

A CR structure on M is a subbundle $T_{1,0} \subset \mathbf{C}T$ such that (i) $T_{1,0} \cap \overline{T_{1,0}} = \{0\}$, (ii) codim $(T_{1,0} \otimes \overline{T_{1,0}}) = 1$, (iii) if X and Y are smooth sections of $T_{1,0}$ then $[X, Y] = XY - YX$ is a section $T_{1,0}$. We set $T_{0,1} = \overline{T_{1,0}}$. If M has a CR structure it is called a CR manifold. An example of a CR manifold is a real hypersurface M in a complex manifold M', $M \subset M'$. Define $T_{1,0}(M) = \mathbf{C}T(M) \cap T_{1,0}(M')$ where $T_{1,0}(M')$ is the holomorphic tangent bundle of M'.

If M is a CR manifold set $T^{1,0}$ (resp., $T^{0,1}$) to be the dual space to $T_{1,0}$ (resp., $T_{0,1}$). Let $\Lambda^{p,q}$ be the space of C^∞ sections of $\Lambda^p T^{1,0} \otimes \Lambda^q T^{0,1}$. Define the operator $\bar\partial_b \colon \Lambda^{p,q} \to \Lambda^{p,q+1}$ as follows: Let $\phi \in \Lambda^{p,q}$ and let $X_1, \ldots, X_p$ (resp., $Y_1, \ldots, Y_{q+1}$) be sections of $T_{1,0}$ (resp., $T_{0,1}$). Then

$$<\bar{\partial}_b\phi; (X_1 \wedge \dots \wedge X_p) \otimes (Y_1 \wedge \dots \wedge Y_{q+1})>$$
$$= (q+1)^{-1} \sum_{j=1}^{q+1} (-1)^{j+1} Y_j <\phi; (X_1 \wedge \dots \wedge X_p) \otimes (Y_1 \wedge \dots \hat{Y}_j \dots \wedge Y_{q+1})>$$
$$+ (q+1)^{-1} \sum_{i<j} (-1)^{i+j} <\phi; (X_1 \wedge \dots \wedge X_p) \otimes ([Y_i, Y_j] \wedge Y_1 \wedge \dots \hat{Y}_i \dots \hat{Y}_j \dots \wedge Y_{q+1})>.$$

The ^ symbol over a section means as usual that it is deleted from the expression. One can show that

i) $\bar{\partial}_b^2 = 0$,

ii) $\bar{\partial}_b(\phi \wedge \psi) = (\bar{\partial}_b\phi) \wedge \psi + (-1)^p\phi \wedge \bar{\partial}_b\psi$ for $\phi \in \Lambda^{0,p}$,

iii) $<\bar{\partial}_b f, Y> = Yf$ for $f \in \Lambda^{0,0}$ and Y a section of $T_{0,1}$.

See [6] for details.

The Heisenberg group, H, is a Lie group with a natural CR structure. The manifold is $\mathbf{C}^n \times \mathbf{R}$. Let (z, t), $(z', t') \in \mathbf{C}^n \times \mathbf{R} = H$. The group law is defined by

$$(z, t) \cdot (z', t') = \big(z + z', t + t' + 2\,\mathrm{Im}(z \cdot z')\big)$$

where $z \cdot z' = \sum_{j=1}^{n} z_j \bar{z}'_j$. The identity element is $(0, 0)$ and $(z, t)^{-1} = (-z, -t)$.

Sometimes we will set $u = (z, t)$.

For $j = 1, \dots, n$ if we set $z_j = x_j + iy_j$, the mapping

$$(z, t) \to (x_1, \dots, x_n, y_1, \dots, y_n, t)$$

defines a C^∞ coordinate system on H. The left invariant vector fields (i.e., the elements of the Lie algebra) are $\mathbf{R}$-linear combinations of

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}. \tag{2}$$

They satisfy the following commutation relations:

$$[X_j, T] = [Y_j, T] = [X_j, X_k] = [Y_j, Y_k] = 0, \tag{3}$$

$$[X_j, Y_k] = -4\delta_{jk}T. \tag{4}$$

Let $\mathbf{C}T$ be the complexified tangent bundle of H. Define

$$Z_j = \frac{1}{2}(X_j - iY_j) = \frac{\partial}{\partial z_j} + i\bar{z}_j\frac{\partial}{\partial t}. \tag{5}$$

Then Z_j, $\bar{Z}_j$, and T form a basis (over $\mathbf{R}$) of the space of left invariant complex tangent vector fields. In particular, they form a global frame of $\mathbf{C}T$. From (3), (4) and (5) we easily see that

$$[Z_j, Z_k] = [\bar{Z}_j, \bar{Z}_k] = [Z_j, T] = [\bar{Z}_j, T] = 0 \qquad (6)$$
$$[Z_j, \bar{Z}_k] = -2i\delta_{jk} T .$$

Let $T_{1,0}$ (resp., $T_{0,1}$) be the subbundle of $\mathbf{C}T$ spanned by the Z_j's (resp., $\bar{Z}_j$'s). Then

$$\bar{T}_{1,0} = T_{0,1}$$
$$T_{1,0} \cap T_{0,1} = \{0\}$$
$$\operatorname{codim}(T_{1,0} \otimes T_{0,1}) = 1 .$$

Also, if V_1, V_2 are sections of $T^{1,0}$ we can write $V_i = \sum_{j=1}^{n} f_{ij} Z_j$, $i = 1, 2$ where the f_{ij} are C^∞ functions on H. Then by (6),

$$[V_1, V_2] = \sum_{k=1}^{n} \left(\sum_{j=1}^{n} (f_{1j} Z_j f_{2k} - f_{2j} Z_j f_{1k}) \right) Z_k .$$

So, the splitting of $\mathbf{C}T$ defines a CR structure on H.

Impose the left invariant Hermitian metric on $\mathbf{C}T$ which makes the Z's, $\bar{Z}$'s and T an orthonormal frame. Let ω^j and τ be dual to Z_j and T, respectively. Then ω^j, $\bar{\omega}^j$ and τ form an orthonormal frame for $\mathbf{C}T^*$. The volume element on H is

$$du = 2^n dx_1 \wedge \dots \wedge dx_n \wedge dy_1 \wedge \dots \wedge dy_n \wedge dt . \qquad (7)$$

Since H is nilpotent and, hence, unimodular, the volume element is both left and right invariant. One can also verify this directly.

Let $J = (j_1, \dots, j_q)$ be a multi-index with $1 \leqslant j_i \leqslant n$, $i = 1, \dots, q$. Define $|J| = q$ and $\bar{\omega}^J = \bar{\omega}^{j_1} \wedge \dots \wedge \bar{\omega}^{j_q}$. If $\phi \in \Lambda^{0,q}(H)$ we may write $\phi = \sum_{|J|=q} \phi_J \bar{\omega}^J$ where ϕ_J is a C^∞ function from H to $\mathbf{C}$. Let

$$\bar{\omega}^j \lrcorner \bar{\omega}^J = (-1)^k \bar{\omega}^{j_1} \wedge \dots \wedge \widehat{\bar{\omega}^{j_k}} \wedge \dots \wedge \bar{\omega}^{j_q} \quad \text{if} \quad j = j_k \quad \text{and} \quad \bar{\omega}^j \lrcorner \bar{\omega}^J = 0$$

otherwise. Folland and Stein prove that for $\phi \in \Lambda^{0,q}$

i) $\bar{\partial}_b \phi = \sum_{|J|=q} \sum_{j=1}^{n} \bar{Z}_j \phi_J \bar{\omega}^j \wedge \bar{\omega}^J$,

(8) ii) $\bar{\partial}_b^* \phi = - \sum_{|J|=q} \sum_{j=1}^{n} Z_j \phi_J \bar{\omega}^j \lrcorner \, \bar{\omega}^J ,$

iii) $\Box_b \phi = \sum_{|J|=q} \left(-\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T \right) \phi_J \bar{\omega}^J .$

Define the function

$$\Phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}} .$$

Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. For an appropriate constant, c_q, define

$$K_q \phi(v) = c_q \sum_{|J|=q} \left(\int_H \phi_J(u) \Phi_{n-2q}(u^{-1}v) du \right) \bar{\omega}^J . \tag{9}$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then* $\Box_b K_q \phi = K_q \Box_b \phi = \phi$.

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then*

i) $H\phi = 0$ *where* H *is the orthogonal projection onto the kernel of* $\Box_b$.

ii) $\phi = \bar{\partial}_b \bar{\partial}_b^* K_q \phi + \bar{\partial}_b^* \bar{\partial}_b K_q \phi$.

We also prove

THEOREM 3. *If* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$ *and if* $\bar{\partial}_b \phi = 0$ *then* $\psi = \bar{\partial}_b^* K_q \phi$ *satisfies* $\bar{\partial}_b \psi = \phi$.

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, $\mathfrak{n}$, is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^{r} \mathfrak{n}_i$ where the $\mathfrak{n}_i$ satisfy

i) $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ if $i + j \leqslant r$,

ii) $[\mathfrak{n}_i, \mathfrak{n}_j] = 0$ if $i + j > r$.

Let $n = \dim \mathfrak{n}$. Define the homogeneous dimension to be $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. If $\mathfrak{n}$ is a graded algebra and if $\mathfrak{n}_1$ generates $\mathfrak{n}$ then $\mathfrak{n}$ is called a stratified algebra. A Lie group is called a stratified group if its Lie algebra is a stratified algebra. For a given stratified algebra $\mathfrak{n}$ we will restrict our attention to the simply connected group associated to it.

The Heisenberg group is a simply connected stratified group. In fact, identifying the Lie algebra with the left invariant vector fields, we may take $\mathfrak{n}_1$ to be the span of the X's and Y's and $\mathfrak{n}_2$ to be the span of T. By (3) and (4) we see that $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$ and $[\mathfrak{n}_1, \mathfrak{n}_2] = [\mathfrak{n}_2, \mathfrak{n}_2] = 0$.

Any graded nilpotent group has a natural family of dilations. First we define them on the Lie algebra. Let $X \in \mathfrak{n}$. Then by definition $X = \sum_{j=1}^{r} X_j$ where $X_j \in \mathfrak{n}_j$. For $s > 0$ set $\delta_s(X) = \sum_{j=1}^{r} s^j X_j$. Because $\mathfrak{n}$ is nilpotent the exponential map is globally defined. Suppose $x \in N$ and $x = \exp(X)$ for $X \in \mathfrak{n}$. Define $\delta_s(x) = \exp(\delta_s X)$. Suppose we are given an inner product on $\mathfrak{n}$ such that $\mathfrak{n}_i \perp \mathfrak{n}_j$ for all $i \neq j$. Let $\| X \|$ be the length defined by the inner product. Suppose $x = \exp(X)$ where $X = \sum_{j=1}^{r} X_j$, $X_j \in \mathfrak{n}_j$. Then define the homogeneous norm function to be

$$| x | = \left(\sum_{j=1}^{r} \| X_j \|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}.$$

Then (i) $| x | = 0$ if and only if $x = 0$, (ii) $x \to | x |$ is continuous on N and C^∞ on $N - \{0\}$, (iii) $| \delta_s x | = s | x |$.

On the Heisenberg group, $\delta_s((z, t)) = (sz, s^2 t)$ and $| z | = (|z|^4 + t^2)^{\frac{1}{4}}$.

Recall that the homogeneous dimension is $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. Let f be a function on N. We say f is homogeneous of degree p if $f(\delta_s(x)) = s^p f(x)$. If $-Q < p$ then such an f is in L^k_{loc} for $1 \leqslant k < \infty$. A distribution F is called homogeneous of degree p if

$$\langle F, s^{-Q} g(\delta_{s^{-1}} x) \rangle = s^p \langle F, g \rangle$$

where $g \in C_c^\infty(N)$ and $\langle F, g \rangle$ is the pairing of $C_c^\infty(N)$ with its dual, $D'(N)$. A differential operator L (acting on functions) is homogeneous of degree p if $L(f \cdot \delta_s) = s^p (Lf) \circ \delta_s$. Observe that if f is a homogeneous function of degree p and if L is a homogeneous differential operator of degree p' then Lf is a homogeneous function of degree $p - p'$.

Let $X_{i,1}, \ldots, X_{i,\dim(\mathfrak{n}_i)}$ be an orthonormal basis of $\mathfrak{n}_i$ with respect to our inner product. Since $\mathfrak{n}_i \perp \mathfrak{n}_j$ for $i \neq j$ the set

$$\{X_{i,j} : 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \dim(\mathfrak{n}_i)\}$$

is an orthonormal basis of $\mathfrak{n}$. Define the global coordinate chart on N by

$$(x_{ij}) \to \Sigma x_{ij} X_{ij} \to \exp(\Sigma x_{ij} X_{ij}) . \tag{10}$$

This identifies N with $\mathbf{R}^n$ as a manifold.

Let m_1, m_2 and m_3 be positive integers. For $i = 1, 2, 3$ define $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$ to be the trivial bundle over $N = \mathbf{R}^n$ with fiber $\mathbf{F}^{m_i}$. Consider the differential complex (1). We know that each D_i can be expressed as an $m_{i+1} \times m_i$ matrix of differential operators on functions, $i = 1, 2$. If each entry is homogeneous of degree p we say D_i is a homogeneous differential operator of degree p. If each entry is left-invariant we say D_i is a left-invariant differential operator.

On our prototype, the Heisenberg group, we have the left-invariant metric which makes the Z's, $\bar{Z}$'s, and T into an orthonormal basis. Let $\bar{\omega}_1, \ldots, \bar{\omega}_n$ be a basis for $T^{0,1}$ which is dual to $\bar{Z}_1, \ldots, \bar{Z}_n$. Then

$$\{\bar{\omega}^J : J = (j_1, \ldots, j_q), 1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant n\}$$

is a global orthonormal basis of $\Lambda^{0,q}$ for each q. So $\Lambda^{0,q}$ is a trivial bundle over $H \approx \mathbf{R}^{2n+1}$, and we may identify sections of $\Lambda^{0,q}$ with $C^\infty(\mathbf{R}^{2n+1}, \mathbf{C}^m)$ where $m = n!/q!(n-q)!$. By (8(iii)) the operator $\Box_b : \Lambda^{0,q} \to \Lambda^{0,q}$ is given by the matrix $(\delta_{ij} L)_{1 \leqslant i,j \leqslant m}$ where $L = -\frac{1}{2} \sum_{k=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T$. L is left-invariant and homogeneous of degree 2. So, $\Box_b$ is left-invariant and homogeneous of degree 2. Similarly, $K_q \phi$ defined by (9) can be written as

$$K_q \phi = \int_H c_q \, \Phi_{n-2q}(u^{-1}v) I \phi \, du$$

where $\phi \in \Lambda_c^{0,q}$ is a $q \times 1$ column vector and I is the $q \times q$ identity matrix. Note that Φ_{n-2q} is a homogeneous function of degree $-2n$. This example motivates the following definition of a homogeneous convolution operator.

Return to N, our stratified Lie group with global coordinates defined by (10). Let $k : N \to \mathrm{Mat}(m' \times m, \mathbf{F})$ be a mapping of N into the space of $m' \times m$ matrices with entries in $\mathbf{F}$. Given $f \in C_c^\infty(\mathbf{F}^m)$ and $x, y \in N$ the product $k(y^{-1}x) f(y)$ is an $m' \times 1$ column vector. We set

(11) $$Kf(x) = \int_N k(y^{-1}x)f(y)dy.$$

The measure, dy, is the Haar measure on N. Under suitable restrictions on k the integral exists. The operator K is called a convolution operator with kernel k. If each entry of k is smooth away from 0 and homogeneous of degree $-Q+p$, $0<p<Q$, we say that K is a homogeneous convolution operator of type p. As we mentioned before, a homogeneous function is in L^p_{loc} so the integral in (11) exists for $f \in C_c^\infty(\mathbf{F}^m)$.

Suppose k is homogeneous of degree $-Q$ and for each entry

$$k_{ij},\ 1 \leqslant i \leqslant m',\ 1 \leqslant j \leqslant m,$$

we have

(12) $$\int_{a \leqslant |x| \leqslant b} k_{ij}(x)dx = 0$$

for all a and b. We say an operator K is of type 0 if for some constant c we have

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leqslant |y| \leqslant 1/\varepsilon} k(y^{-1}x)f(y)dy + cf(0) \quad \text{for all} \quad f \in C_c^\infty(\mathbf{F}^m)$$

where k satisfies (12). We refer the reader to Folland [9] or Rothschild and Stein [16] for details.

To study the continuity properties of these operators we define L^p spaces and Sobolev-type spaces of sections from N to $\mathbf{F}^m$. Let $\|\ \|_{L^p}$ denote the usual L^p norm on functions. Let $f \in C_c^\infty(\mathbf{F}^m)$ and let f_i, $i = 1, \ldots, m$ be the components of f. Define the norm

$$\| f \|_{L^p(F^m)} = \left(\sum_{i=1}^m \| f_i \|_{L^p}^p \right)^{1/p}.$$

Let $L^p(\mathbf{F}^m)$ be the completion of $C_c^\infty(\mathbf{F}^m)$ under this norm.

Let $\{X_{1,1}, \ldots, X_{1,d}\}$ be the orthonormal basis of $\mathfrak{n}_1$, with $d = \dim(\mathfrak{n}_1)$. For brevity, we will drop reference to the first subscript. Let J be a multi-index, $J = (j_1, j_2, \ldots, j_q)$ with $1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant d$. Define $|J| = q$ and define $X_J = X_{j_1}X_{j_2} \ldots X_{j_q}$. Define $S_q^p(\mathbf{F}^m)$ to be the closure of $C_c^\infty(\mathbf{F}^m)$ under the norm

$$\| f \|_{S_q^p(F^m)} = \left(\| f \|_{L^p(F^m)}^p + \sum_{i=1}^m \sum_{|J| \leqslant q} \| X_J f_i \|_{L^p}^p \right)^{1/p}.$$

A modification of a theorem by Folland [9] yields

THEOREM 4. (i) *Let* K *be a convolution operator of type* r *for* $r > 0$. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $L^p(\mathbf{F}^m)$ *to* $L^q(\mathbf{F}^{m'})$ *where* $1 < p < Q/r$ *and* $q^{-1} = p^{-1} - r/Q$. (ii) *Let* K *be a convolution operator of type* 0. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $S_k^p(\mathbf{F}^m)$ *to* $S_k^p(\mathbf{F}^{m'})$.

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let $D: C^\infty(\mathbf{F}^{m'}) \to C^\infty(\mathbf{F}^{m''})$ be a left-invariant homogeneous differential operator of degree 1 and let K be a homogeneous convolution operator of type r, with $r \geqslant 1$. Then DK is a homogeneous convolution operator of type $r - 1$. Moreover, if $r > 1$ the kernel of DK is given by $Dk(x)$.

4. THE HODGE DECOMPOSITION

Consider the complex (1) where $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$. Assume that each of the D_i is a first order, left-invariant operator, homogeneous of degree 1. So each entry of D_i is of the form $\sum_{j=1}^{d} a_j X_{1,j}$ where a_j is constant. Construct the Laplacian, Δ, with respect to the euclidian inner products on $\mathbf{F}^{m_i}$, $i = 1, 2, 3$. Assume there exists a homogeneous convolution operator of type 2, K, which inverts Δ. If $f \in C_c^\infty(\mathbf{F}^{m_2})$ then $f(x) = \Delta K f(x) = K \Delta f(x)$.

THEOREM 5. *Let* $f \in S_2^2(\mathbf{F}^{m_2})$. *As distributions,* $\Delta f = 0$ *if and only if* $f = 0$.

Proof. Obviously, if $f = 0$ then $\Delta f = 0$.

Assume $\Delta f = 0$. Let $\{f_j\}$ be a sequence in $C_c^\infty(\mathbf{F}^{m_2})$ such that $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$. Then $f_j \to f$ in the sense of distributions. Moreover, $\Delta f_j \to \Delta f = 0$ in $L^2(\mathbf{F}^{m_2})$. Let $g \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$<f, g> = \lim_{j\to\infty} <f_j, g> = \lim_{j\to\infty} <f_j, \Delta K g> = \lim_{j\to\infty} <\Delta f_j, Kg> .$$

Because $g \in C_c^\infty(\mathbf{F}^{m_2})$ it is in L^p where $p = 2Q/(Q+4)$. Therefore, by Theorem 4(i), $Kg \in L^q$ where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}) .$$

For $Q \geqslant 5$, $1 < p < q < \infty$. So

$$| <f, g> | = \lim_{j\to\infty} | <\Delta f_j, Kg> | \leqslant \lim_{j\to\infty} \| \Delta f_j \|_{L^2(\mathbf{F}^{m_2})} \| Kg \|_{L^2(\mathbf{F}^{m_2})} = 0 .$$

So, as a distribution, $f = 0$. This proves the theorem.

We have shown that the only harmonic element in $S_2^2(\mathbf{F}^{m_2})$ is the zero element. Let $f \in S_2^2(\mathbf{F}^{m_2})$ and let $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$ with $f_j \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$f = \lim_{j\to\infty} \Delta K f_j = \lim_{j\to\infty} D_1 D_1^* K f + \lim_{j\to\infty} D_2^* D_2 K f = D_1 D_1^* K f + D_2^* D_2 K f .$$

To complete the Hodge decomposition we must prove that

$$D_1 D_1^* K f \perp D_2^* D_2 K f .$$

We need the following notation. Let $D(R) = \{x \in N : |x| < R\}$ and $S(R) = \{x \in N : |x| = R\}$. Endow each set with the left-invariant metric induced by N. The metric gives rise to the corresponding volume elements which, in the case of $D(R)$, is the restriction of dx. Let $d\mu_R$ denote the volume element on $S(R)$. For $f, g \in C_c^\infty(D(R), \mathbf{F}^{m_i})$ define

$$(f, g)_{D(R), i} = \int_{D(R)} (f(x), g(x))_{i,x} dx$$

where $(\ ,\ )_{i,x}$ is the metric on $\mathbf{F}^{m_i}$. Similarly for $f, g \in C^\infty(S(R), \mathbf{F}^{m_i})$ define

$$(f, g)_{S(R), i} = \int_{S(R)} (f(x), g(x))_{i,x} d\mu_R(x) .$$

By restriction, any element $f \in C^\infty(N, \mathbf{F}^{m_i})$ gives rise to an element of $C^\infty(S(R), \mathbf{F}^{m_i})$ or $C^\infty(D(R), \mathbf{F}^{m_i})$. In our notation, we will not distinguish f from its restrictions.

We will be integrating by parts on $D(R)$ which will involve a boundary integral on $S(R)$. To that end we define the symbol of our differential operators. Define $\xi(x) = |x| - R$ and let $g \in C^1(N, \mathbf{F}^{m_i})$. Let $x \in S(R)$. Then $\xi(x) = 0$. The symbol of D_i at $x \in S(R)$ acting on $d\xi$ and on g is given by

$$\sigma(D_i, d\xi) g(x) = D_i(\xi g)(x) .$$

The integration by parts formula is

$$(D_i g, f)_{D(R), i+1} = (g, D_i^* f)_{D(R), i} + (\sigma(D_i, d\xi) g, f)_{S(R), i} . \tag{13}$$

THEOREM 6. *Assume* $f \in L^2(\mathbf{F}^{m_2}) \cap L^q(\mathbf{F}^{m_2})$ *where* $q = Q/Q + 2$. *Then* $f = D_1 D_1^* K f + D_2^* D_2 K f$ *and* $D_1 D_1^* K f \perp D_2^* D_2 K f$.

Proof. We have already seen that $f = D_1 D_1^* K f + D_2^* D_2 K f$. To prove the orthogonality we restrict our attention to $D(R)$ for R large.

For brevity let $h = (D_1 D_1^* Kf, D_2^* D_2 Kf)_2$. Also, let

$$h(R) = (D_1 D_1^* Kf, D_2^* D_2 Kf)_{D(R),2}.$$

Then $\lim_{R\to\infty} h(R) = h$. Note that $D_1 D_1^* K$ and $D_2^* D_2 K$ are type 0 operators. Since $f \in L^2(\mathbf{F}^{m_2})$ by theorem 4 (ii) we know that h is defined. Furthermore $h(R)$ is bounded for all R by $\| D_1 D_1^* Kf \|_{L^2(R^{m_2})} \| D_2^* D_2 Kf \|_{L^2(R^{m_2})}$.

We can compute $h(R)$ as follows:

$$h(R) = \big(D_1 D_1^* Kf, D_2^* D_2(Kf)\big)_{D(R),2} = (D_2 D_1 D_1^* Kf, D_2 Kf)_{D(R),3} + \big(D_1 D_1^* Kf, \sigma(D_2^*, d(|x|)) D_2 Kf\big)_{S(R),2}$$

by (13). We now prove a sequence of lemmas.

LEMMA 1. *$h(R)$ is continuous.*

Proof. This follows from Lebesgue's dominated convergence theorem.

LEMMA 2. *Let $R \geqslant 1, x \in N$ and $|x| = R$. Then*

$$|\sigma(D_2^*, d|x|) g(x)| \leqslant C |g(x)|$$

where C is a constant independent of g.

Proof. Recall that $X_1, \ldots, X_d$ is our orthonormal basis for $\mathfrak{n}_1$ where $d = \dim(\mathfrak{n}_1)$. The entries of D_2^* are linear combinations of the X_i, $i = 1, \ldots, d$ with coefficients in $\mathbf{F}$. Let D_{ij} be the i,j entry, $1 \leqslant i \leqslant m_2$, $1 \leqslant j \leqslant m_3$. Then $D_{ij} = \sum_{k=1}^{d} C_{ij}^k X_k$. Thus, for $x \in S(R)$

$$\begin{aligned}
|\sigma(D_2^*, d|x|) g(x)| &\leqslant C \sum_{i=1}^{m_2} \Big| \sum_{j=1}^{m_3} D_{ij}\big((|x|-R) g_j(x)\big) \Big| \\
&\leqslant C \sum_{i,j,k} |C_{ij}^k X_k\big((|x|-R) g_j(x)\big)| \\
&\leqslant C \sum_{jk} |(X_k|x|) g_j(x)| \qquad (\text{since } |x| = R) \\
&\leqslant C\big(\max_k (X_k|x|)\big) |g_j(x)|.
\end{aligned}$$

We must show $X_k |x|$ is bounded. But $|x|$ is C^∞ away from the origin and homogeneous of degree 1. So $X_k |x|$ is homogeneous of degree 0. Thus, it is determined by its values on $\{|x| = 1\}$. It is C^∞ on this set and, therefore, bounded. This proves the lemma.

Since $h(R) \to h$ as $R \to \infty$ we have

LEMMA 3. *For* $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = h.$

We continue with the proof of our theorem. By the preceding lemma it suffices for us to prove that $\lim_{r\to\infty} \frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0$. But,

$$\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \tag{14}$$

$$= \frac{1}{2\varepsilon}\int_{r-\varepsilon\leqslant|x|\leqslant r+\varepsilon} (D_1D_1^*Kf, \sigma(D_2^*, d|x|)D_2Kf)_x \,\|\, d\,|\,x\,|\,\|\, dx$$

because $d\mu_R dR = \|\, d\,|\,x\,|\,\|\, dx$. We claim that $\|\, d\,|\,x\,|\,\|$ is bounded. Let ω^{ij} be dual to X_{ij} where X_{ij} is our orthonormal basis. Then $d\,|\,x\,| = \sum_{i,j}(X_{ij}|x|)\omega^{ij}$. Since $|\,x\,|$ is homogeneous of degree 1 and X_{ij} is homogeneous of degree i we have $X_{ij}\,|\,x\,|$ is homogeneous of degree $1 - i$. Hence, for $|\,x\,| \geqslant 1$ each $X_{ij}\,|\,x\,|$ is bounded. So $\|\, d\,|\,x\,|\,\|$ is bounded.

By assumption, $f \in L^q(\mathbf{F}^{m_2})$, $q = Q/Q + 2$ and we know that D_2K is type 1. By Theorem 4(i) we know that $D_2Kf \in L^2(\mathbf{F}^{m_2})$. Thus, by Lemma 2 $|\,\chi_r\sigma(D_2^*, d|x|)D_2Kf\,| \leqslant C\,|\,\chi_r D_2Kf\,|$ where χ_r is the characteristic function of $\{r - \varepsilon \leqslant |\,x\,| \leqslant r + \varepsilon\}$. We conclude that $\chi_r\sigma(D_2^*, d|x|)D_2Kf \in L^2(\mathbf{F}^{m_2})$. By the Schwarz inequality and the fact that $\|\, d\,|\,x\,|\,\|$ is bounded, we get, from (14)

$$\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \leqslant \frac{c}{2\varepsilon}\,\|\,\chi_r D_1D_1^*Kf\,\|_{L^2(\mathbf{F}^{m_2})}\,\|\,\chi_r D_2Kf\,\|_{L^2(\mathbf{F}^{m_3})}\,.$$

As $r \to \infty$, both $\|\,\chi_r D_1D_1^*Kf\,\|_{L^2(\mathbf{F}^{m_2})}$ and $\|\,\chi_r D_2Kf\,\|_{L^2(\mathbf{F}^{m_3})}$ tend to 0. So $h = \lim_{r\to\infty}\frac{1}{2\varepsilon}\int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0$. This proves the theorem.

This theorem together with Theorem 5 proves the Hodge decomposition. A similar argument gives the solution to the problem of finding g such that $D_1g = f$ for a given f.

THEOREM 7. *Let* $f \in L^2(\mathbf{F}^{m_2}) \cap L^q(\mathbf{F}^{m_2})$ *with* $q = Q/Q + 2$. *Suppose* $D_2f = 0$. *Then there exists* $g \in L^2(\mathbf{F}^{m_1})$ *such that* $D_1g = f$.

Proof. We have $f = D_1D_1^*Kf + D_2^*D_2Kf$. It suffices to prove $(f, D_2^*D_2Kf) = 0$ because this implies $D_2^*D_2Kf = 0$ since

$$D_1D_1^*Kf \perp D_2^*D_2Kf\,.$$

We may set $g = D_1^* Kf$. Using the same notation as in the preceding theorem we have

$$(f, D_2^* D_2 Kf)_2 = \lim_{R \to \infty} (f, D_2^* D_2 Kf)_{D(R), 2}$$

$$= \lim_{R \to \infty} \left((D_2 f, D_2 Kf)_{D(R), 3} + (f, \sigma(D_2^*, d|x|) D_2 Kf)_{S(R), 2}\right)$$

$$= \lim_{R \to \infty} (f, \sigma(D_2^*, d|x|) D_2 Kf)_{S(R), 2} \quad (\text{since } D_2 f = 0).$$

The same argument as in Theorem 6 proves that the limit is zero. This proves the theorem.

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