3. Differential complexes on stratified groups

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **16.07.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.
Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.
Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

Ein Dienst der *ETH-Bibliothek*
ETH Zürich, Rämistrasse 101, 8092 Zürich, Schweiz, www.library.ethz.ch

http://www.e-periodica.ch

(8) ii) $\bar{\partial}_b^* \phi = -\sum_{|J|=q} \sum_{j=1}^{n} Z_j \phi_J \bar{\omega}^j \lrcorner\, \bar{\omega}^J$,

iii) $\square_b \phi = \sum_{|J|=q} \left(-\frac{1}{2} \sum_{j=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T \right) \phi_J \bar{\omega}^J$.

Define the function

$$\Phi_\alpha(z, t) = (|z|^2 - it)^{-\frac{n+\alpha}{2}} (|z|^2 + it)^{-\frac{n-\alpha}{2}} .$$

Let $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. For an appropriate constant, c_q, define

(9) $$K_q \phi(v) = c_q \sum_{|J|=q} \left(\int_H \phi_J(u) \Phi_{n-2q}(u^{-1}v) du \right) \bar{\omega}^J .$$

Folland and Stein prove that for the appropriate c_q

THEOREM 1. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then* $\square_b K_q \phi = K_q \square_b \phi = \phi$.

In [4] we prove a stronger version of the following Hodge decomposition theorem.

THEOREM 2. *Let* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$. *Then*

i) $H\phi = 0$ *where* H *is the orthogonal projection onto the kernel of* $\square_b$.

ii) $\phi = \bar{\partial}_b \bar{\partial}_b^* K_q \phi + \bar{\partial}_b^* \bar{\partial}_b K_q \phi$.

We also prove

THEOREM 3. *If* $\phi \in \Lambda_c^{0,q}$, $q \neq 0, n$ *and if* $\bar{\partial}_b \phi = 0$ *then* $\psi = \bar{\partial}_b^* K_q \phi$ *satisfies* $\bar{\partial}_b \psi = \phi$.

These two theorems are special cases of theorems 6 and 7 proven in section 4.

3. DIFFERENTIAL COMPLEXES ON STRATIFIED GROUPS

We study a class of nilpotent Lie groups which we describe in terms of their Lie algebras. A graded Lie algebra, $\mathfrak{n}$, is a finite dimensional nilpotent algebra which has a direct sum decomposition, $\mathfrak{n} = \bigoplus_{i=1}^{r} \mathfrak{n}_i$ where the $\mathfrak{n}_i$ satisfy

i) $[\mathfrak{n}_i, \mathfrak{n}_j] \subseteq \mathfrak{n}_{i+j}$ if $i + j \leqslant r$,

ii) $[\mathfrak{n}_i, \mathfrak{n}_j] = 0$ if $i + j > r$.

Let $n = \dim \mathfrak{n}$. Define the homogeneous dimension to be $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. If $\mathfrak{n}$ is a graded algebra and if $\mathfrak{n}_1$ generates $\mathfrak{n}$ then $\mathfrak{n}$ is called a stratified algebra. A Lie group is called a stratified group if its Lie algebra is a stratified algebra. For a given stratified algebra $\mathfrak{n}$ we will restrict our attention to the simply connected group associated to it.

The Heisenberg group is a simply connected stratified group. In fact, identifying the Lie algebra with the left invariant vector fields, we may take $\mathfrak{n}_1$ to be the span of the X's and Y's and $\mathfrak{n}_2$ to be the span of T. By (3) and (4) we see that $[\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2$ and $[\mathfrak{n}_1, \mathfrak{n}_2] = [\mathfrak{n}_2, \mathfrak{n}_2] = 0$.

Any graded nilpotent group has a natural family of dilations. First we define them on the Lie algebra. Let $X \in \mathfrak{n}$. Then by definition $X = \sum_{j=1}^{r} X_j$ where $X_j \in \mathfrak{n}_j$. For $s > 0$ set $\delta_s(X) = \sum_{j=1}^{r} s^j X_j$. Because $\mathfrak{n}$ is nilpotent the exponential map is globally defined. Suppose $x \in N$ and $x = \exp(X)$ for $X \in \mathfrak{n}$. Define $\delta_s(x) = \exp(\delta_s X)$. Suppose we are given an inner product on $\mathfrak{n}$ such that $\mathfrak{n}_i \perp \mathfrak{n}_j$ for all $i \neq j$. Let $\| X \|$ be the length defined by the inner product. Suppose $x = \exp(X)$ where $X = \sum_{j=1}^{r} X_j$, $X_j \in \mathfrak{n}_j$. Then define the homogeneous norm function to be

$$| x | = \left(\sum_{j=1}^{r} \| X_j \|^{\frac{2r!}{j}} \right)^{\frac{1}{2r!}}.$$

Then (i) $| x | = 0$ if and only if $x = 0$, (ii) $x \to | x |$ is continuous on N and C^∞ on $N - \{0\}$, (iii) $| \delta_s x | = s | x |$.

On the Heisenberg group, $\delta_s((z, t)) = (sz, s^2 t)$ and $| z | = (|z|^4 + t^2)^{\frac{1}{4}}$.

Recall that the homogeneous dimension is $Q = \sum_{j=1}^{r} j \dim(\mathfrak{n}_j)$. Let f be a function on N. We say f is homogeneous of degree p if $f(\delta_s(x)) = s^p f(x)$. If $-Q < p$ then such an f is in L^k_{loc} for $1 \leqslant k < \infty$. A distribution F is called homogeneous of degree p if

$$< F, s^{-Q} g(\delta_{s^{-1}} x) > = s^p < F, g >$$

where $g \in C_c^\infty(N)$ and $< F, g >$ is the pairing of $C_c^\infty(N)$ with its dual, $D'(N)$. A differential operator L (acting on functions) is homogeneous of degree p if $L(f \cdot \delta_s) = s^p (Lf) \circ \delta_s$. Observe that if f is a homogeneous function of degree p and if L is a homogeneous differential operator of degree p' then Lf is a homogeneous function of degree $p - p'$.

Let $X_{i,1}, \ldots, X_{i,\dim(\mathfrak{n}_i)}$ be an orthonormal basis of $\mathfrak{n}_i$ with respect to our inner product. Since $\mathfrak{n}_i \perp \mathfrak{n}_j$ for $i \neq j$ the set

$$\{X_{i,j}: 1 \leqslant i \leqslant r, 1 \leqslant j \leqslant \dim(\mathfrak{n}_i)\}$$

is an orthonormal basis of $\mathfrak{n}$. Define the global coordinate chart on N by

$$(x_{ij}) \to \Sigma x_{ij} X_{ij} \to \exp(\Sigma x_{ij} X_{ij}) . \tag{10}$$

This identifies N with $\mathbf{R}^n$ as a manifold.

Let m_1, m_2 and m_3 be positive integers. For $i = 1, 2, 3$ define $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$ to be the trivial bundle over $N = \mathbf{R}^n$ with fiber $\mathbf{F}^{m_i}$. Consider the differential complex (1). We know that each D_i can be expressed as an $m_{i+1} \times m_i$ matrix of differential operators on functions, $i = 1, 2$. If each entry is homogeneous of degree p we say D_i is a homogeneous differential operator of degree p. If each entry is left-invariant we say D_i is a left-invariant differential operator.

On our prototype, the Heisenberg group, we have the left-invariant metric which makes the Z's, $\bar{Z}$'s, and T into an orthonormal basis. Let $\bar{\omega}_1, \ldots, \bar{\omega}_n$ be a basis for $T^{0,1}$ which is dual to $\bar{Z}_1, \ldots, \bar{Z}_n$. Then

$$\{\bar{\omega}^J: J = (j_1, \ldots, j_q), 1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant n\}$$

is a global orthonormal basis of $\Lambda^{0,q}$ for each q. So $\Lambda^{0,q}$ is a trivial bundle over $H \approx \mathbf{R}^{2n+1}$, and we may identify sections of $\Lambda^{0,q}$ with $C^\infty(\mathbf{R}^{2n+1}, \mathbf{C}^m)$ where $m = n!/q!(n-q)!$. By (8(iii)) the operator $\Box_b: \Lambda^{0,q} \to \Lambda^{0,q}$ is given by the matrix $(\delta_{ij} L)_{1 \leqslant i,j \leqslant m}$ where $L = -\frac{1}{2} \sum_{k=1}^{n} (Z_j \bar{Z}_j + \bar{Z}_j Z_j) + i(n-2q)T$. L is left-invariant and homogeneous of degree 2. So, $\Box_b$ is left-invariant and homogeneous of degree 2. Similarly, $K_q \phi$ defined by (9) can be written as

$$K_q \phi = \int_H c_q \, \Phi_{n-2q}(u^{-1}v) I \phi du$$

where $\phi \in \Lambda_c^{0,q}$ is a $q \times 1$ column vector and I is the $q \times q$ identity matrix. Note that Φ_{n-2q} is a homogeneous function of degree $-2n$. This example motivates the following definition of a homogeneous convolution operator.

Return to N, our stratified Lie group with global coordinates defined by (10). Let $k: N \to \mathrm{Mat}(m' \times m, \mathbf{F})$ be a mapping of N into the space of $m' \times m$ matrices with entries in $\mathbf{F}$. Given $f \in C_c^\infty(\mathbf{F}^m)$ and $x, y \in N$ the product $k(y^{-1}x) f(y)$ is an $m' \times 1$ column vector. We set

(11) $$Kf(x) = \int_N k(y^{-1}x)f(y)dy\,.$$

The measure, dy, is the Haar measure on N. Under suitable restrictions on k the integral exists. The operator K is called a convolution operator with kernel k. If each entry of k is smooth away from 0 and homogeneous of degree $-Q+p$, $0<p<Q$, we say that K is a homogeneous convolution operator of type p. As we mentioned before, a homogeneous function is in L^p_{loc} so the integral in (11) exists for $f \in C_c^\infty(\mathbf{F}^m)$.

Suppose k is homogeneous of degree $-Q$ and for each entry

$$k_{ij},\ 1 \leqslant i \leqslant m',\ 1 \leqslant j \leqslant m,$$

we have

(12) $$\int_{a \leqslant |x| \leqslant b} k_{ij}(x)dx = 0$$

for all a and b. We say an operator K is of type 0 if for some constant c we have

$$Kf(x) = \lim_{\varepsilon \to 0} \int_{\varepsilon \leqslant |y| \leqslant 1/\varepsilon} k(y^{-1}x)f(y)dy + cf(0) \quad \text{for all} \quad f \in C_c^\infty(\mathbf{F}^m)$$

where k satisfies (12). We refer the reader to Folland [9] or Rothschild and Stein [16] for details.

To study the continuity properties of these operators we define L^p spaces and Sobolev-type spaces of sections from N to $\mathbf{F}^m$. Let $\|\ \|_{L^p}$ denote the usual L^p norm on functions. Let $f \in C_c^\infty(\mathbf{F}^m)$ and let f_i, $i = 1, \ldots, m$ be the components of f. Define the norm

$$\| f \|_{L^p(F^m)} = \left(\sum_{i=1}^m \| f_i \|_{L^p}^p \right)^{1/p}.$$

Let $L^p(\mathbf{F}^m)$ be the completion of $C_c^\infty(\mathbf{F}^m)$ under this norm.

Let $\{X_{1,1}, \ldots, X_{1,d}\}$ be the orthonormal basis of $\mathfrak{n}_1$, with $d = \dim(\mathfrak{n}_1)$. For brevity, we will drop reference to the first subscript. Let J be a multi-index, $J = (j_1, j_2, \ldots, j_q)$ with $1 \leqslant j_1 < j_2 < \ldots < j_q \leqslant d$. Define $|J| = q$ and define $X_J = X_{j_1}X_{j_2} \ldots X_{j_q}$. Define $S_q^p(\mathbf{F}^m)$ to be the closure of $C_c^\infty(\mathbf{F}^m)$ under the norm

$$\| f \|_{S_q^p(F^m)} = \left(\| f \|_{L^p(F^m)}^p + \sum_{i=1}^m \sum_{|J| \leqslant q} \| X_J f_i \|_{L^p}^p \right)^{1/p}.$$

A modification of a theorem by Folland [9] yields

THEOREM 4. (i) *Let* K *be a convolution operator of type* r *for* $r > 0$. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $L^p(\mathbf{F}^m)$ *to* $L^q(\mathbf{F}^{m'})$ *where* $1 < p < Q/r$ *and* $q^{-1} = p^{-1} - r/Q$. (ii) *Let* K *be a convolution operator of type* 0. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $S_k^p(\mathbf{F}^m)$ *to* $S_k^p(\mathbf{F}^{m'})$.

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let $D: C^\infty(\mathbf{F}^{m'}) \to C^\infty(\mathbf{F}^{m''})$ be a left-invariant homogeneous differential operator of degree 1 and let K be a homogeneous convolution operator of type r, with $r \geqslant 1$. Then DK is a homogeneous convolution operator of type $r - 1$. Moreover, if $r > 1$ the kernel of DK is given by $Dk(x)$.

4. THE HODGE DECOMPOSITION

Consider the complex (1) where $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$. Assume that each of the D_i is a first order, left-invariant operator, homogeneous of degree 1. So each entry of D_i is of the form $\sum_{j=1}^{d} a_j X_{1,j}$ where a_j is constant. Construct the Laplacian, Δ, with respect to the euclidian inner products on $\mathbf{F}^{m_i}$, $i = 1, 2, 3$. Assume there exists a homogeneous convolution operator of type 2, K, which inverts Δ. If $f \in C_c^\infty(\mathbf{F}^{m_2})$ then $f(x) = \Delta K f(x) = K \Delta f(x)$.

THEOREM 5. *Let* $f \in S_2^2(\mathbf{F}^{m_2})$. *As distributions,* $\Delta f = 0$ *if and only if* $f = 0$.

Proof. Obviously, if $f = 0$ then $\Delta f = 0$.

Assume $\Delta f = 0$. Let $\{f_j\}$ be a sequence in $C_c^\infty(\mathbf{F}^{m_2})$ such that $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$. Then $f_j \to f$ in the sense of distributions. Moreover, $\Delta f_j \to \Delta f = 0$ in $L^2(\mathbf{F}^{m_2})$. Let $g \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$<f, g> = \lim_{j\to\infty} <f_j, g> = \lim_{j\to\infty} <f_j, \Delta K g> = \lim_{j\to\infty} <\Delta f_j, K g> .$$

Because $g \in C_c^\infty(\mathbf{F}^{m_2})$ it is in L^p where $p = 2Q/(Q+4)$. Therefore, by Theorem 4(i), $Kg \in L^q$ where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}) .$$

For $Q \geqslant 5$, $1 < p < q < \infty$. So

$$| <f, g> | = \lim_{j\to\infty} | <\Delta f_j, Kg> | \leqslant \lim_{j\to\infty} \| \Delta f_j \|_{L^2(\mathbf{F}^{m_2})} \| Kg \|_{L^2(\mathbf{F}^{m_2})} = 0 .$$