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Kapitel: 4. The Hodge decomposition

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THEOREM 4. (i) *Let* K *be a convolution operator of type* r *for* $r > 0$. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $L^p(\mathbf{F}^m)$ *to* $L^q(\mathbf{F}^{m'})$ *where* $1 < p < Q/r$ *and* $q^{-1} = p^{-1} - r/Q$. (ii) *Let* K *be a convolution operator of type* 0. *Then* K *extends from* $C_c^\infty(\mathbf{F}^m)$ *to a bounded operator from* $S_k^p(\mathbf{F}^m)$ *to* $S_k^p(\mathbf{F}^{m'})$.

Finally, we mention the interaction between the homogeneous convolution operators and the left-invariant differential operators. Let $D: C^\infty(\mathbf{F}^{m'}) \to C^\infty(\mathbf{F}^{m''})$ be a left-invariant homogeneous differential operator of degree 1 and let K be a homogeneous convolution operator of type r, with $r \geqslant 1$. Then DK is a homogeneous convolution operator of type $r - 1$. Moreover, if $r > 1$ the kernel of DK is given by $Dk(x)$.

4. THE HODGE DECOMPOSITION

Consider the complex (1) where $E_i = \mathbf{R}^n \times \mathbf{F}^{m_i}$. Assume that each of the D_i is a first order, left-invariant operator, homogeneous of degree 1. So each entry of D_i is of the form $\sum_{j=1}^{d} a_j X_{1,j}$ where a_j is constant. Construct the Laplacian, Δ, with respect to the euclidian inner products on $\mathbf{F}^{m_i}$, $i = 1, 2, 3$. Assume there exists a homogeneous convolution operator of type 2, K, which inverts Δ. If $f \in C_c^\infty(\mathbf{F}^{m_2})$ then $f(x) = \Delta K f(x) = K\Delta f(x)$.

THEOREM 5. *Let* $f \in S_2^2(\mathbf{F}^{m_2})$. *As distributions,* $\Delta f = 0$ *if and only if* $f = 0$.

Proof. Obviously, if $f = 0$ then $\Delta f = 0$.

Assume $\Delta f = 0$. Let $\{f_j\}$ be a sequence in $C_c^\infty(\mathbf{F}^{m_2})$ such that $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$. Then $f_j \to f$ in the sense of distributions. Moreover, $\Delta f_j \to \Delta f = 0$ in $L^2(\mathbf{F}^{m_2})$. Let $g \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$<f, g> = \lim_{j\to\infty} <f_j, g> = \lim_{j\to\infty} <f_j, \Delta K g> = \lim_{j\to\infty} <\Delta f_j, Kg> .$$

Because $g \in C_c^\infty(\mathbf{F}^{m_2})$ it is in L^p where $p = 2Q/(Q+4)$. Therefore, by Theorem 4(i), $Kg \in L^q$ where

$$q^{-1} = (Q+4)/2Q - 2/Q = 1/2, \text{ i.e., } Kg \in L^2(\mathbf{F}^{m_2}) .$$

For $Q \geqslant 5$, $1 < p < q < \infty$. So

$$| <f, g> | = \lim_{j\to\infty} | <\Delta f_j, Kg> | \leqslant \lim_{j\to\infty} \| \Delta f_j \|_{L^2(\mathbf{F}^{m_2})} \| Kg \|_{L^2(\mathbf{F}^{m_2})} = 0 .$$

So, as a distribution, $f = 0$. This proves the theorem.

We have shown that the only harmonic element in $S_2^2(\mathbf{F}^{m_2})$ is the zero element. Let $f \in S_2^2(\mathbf{F}^{m_2})$ and let $f_j \to f$ in $S_2^2(\mathbf{F}^{m_2})$ with $f_j \in C_c^\infty(\mathbf{F}^{m_2})$. Then

$$f = \lim_{j\to\infty} \Delta K f_j = \lim_{j\to\infty} D_1 D_1^* K f + \lim_{j\to\infty} D_2^* D_2 K f = D_1 D_1^* K f + D_2^* D_2 K f .$$

To complete the Hodge decomposition we must prove that

$$D_1 D_1^* K f \perp D_2^* D_2 K f .$$

We need the following notation. Let $D(R) = \{x \in N : |x| < R\}$ and $S(R) = \{x \in N : |x| = R\}$. Endow each set with the left-invariant metric induced by N. The metric gives rise to the corresponding volume elements which, in the case of $D(R)$, is the restriction of dx. Let $d\mu_R$ denote the volume element on $S(R)$. For $f, g \in C_c^\infty(D(R), \mathbf{F}^{m_i})$ define

$$(f, g)_{D(R), i} = \int_{D(R)} (f(x), g(x))_{i, x} dx$$

where $(\ ,\ )_{i,x}$ is the metric on $\mathbf{F}^{m_i}$. Similarly for $f, g \in C^\infty(S(R), \mathbf{F}^{m_i})$ define

$$(f, g)_{S(R), i} = \int_{S(R)} (f(x), g(x))_{i, x} d\mu_R(x) .$$

By restriction, any element $f \in C^\infty(N, \mathbf{F}^{m_i})$ gives rise to an element of $C^\infty(S(R), \mathbf{F}^{m_i})$ or $C^\infty(D(R), \mathbf{F}^{m_i})$. In our notation, we will not distinguish f from its restrictions.

We will be integrating by parts on $D(R)$ which will involve a boundary integral on $S(R)$. To that end we define the symbol of our differential operators. Define $\xi(x) = |x| - R$ and let $g \in C^1(N, \mathbf{F}^{m_i})$. Let $x \in S(R)$. Then $\xi(x) = 0$. The symbol of D_i at $x \in S(R)$ acting on $d\xi$ and on g is given by

$$\sigma(D_i, d\xi) g(x) = D_i(\xi g)(x) .$$

The integration by parts formula is

$$(D_i g, f)_{D(R), i+1} = (g, D_i^* f)_{D(R), i} + (\sigma(D_i, d\xi) g, f)_{S(R), i} . \tag{13}$$

THEOREM 6. *Assume* $f \in L^2(\mathbf{F}^{m_2}) \cap L^q(\mathbf{F}^{m_2})$ *where* $q = Q/Q + 2$. *Then* $f = D_1 D_1^* K f + D_2^* D_2 K f$ *and* $D_1 D_1^* K f \perp D_2^* D_2 K f$.

Proof. We have already seen that $f = D_1 D_1^* K f + D_2^* D_2 K f$. To prove the orthogonality we restrict our attention to $D(R)$ for R large.

For brevity let $h = (D_1 D_1^* Kf, D_2^* D_2 Kf)_2$. Also, let

$$h(R) = (D_1 D_1^* Kf, D_2^* D_2 Kf)_{D(R),2}.$$

Then $\lim_{R\to\infty} h(R) = h$. Note that $D_1 D_1^* K$ and $D_2^* D_2 K$ are type 0 operators. Since $f \in L^2(\mathbf{F}^{m_2})$ by theorem 4 (ii) we know that h is defined. Furthermore $h(R)$ is bounded for all R by $\| D_1 D_1^* Kf \|_{L^2(R^{m_2})} \| D_2^* D_2 Kf \|_{L^2(R^{m_2})}$.

We can compute $h(R)$ as follows:

$$\begin{aligned} h(R) = \big(D_1 D_1^* Kf, D_2^* D_2(Kf)\big)_{D(R),2} &= (D_2 D_1 D_1^* Kf, D_2 Kf)_{D(R),3} \\ &+ \big(D_1 D_1^* Kf, \sigma(D_2^*, d(|x|)) D_2 Kf\big)_{S(R),2} \end{aligned}$$

by (13). We now prove a sequence of lemmas.

Lemma 1. *$h(R)$ is continuous.*

Proof. This follows from Lebesgue's dominated convergence theorem.

Lemma 2. *Let $R \geqslant 1, x \in N$ and $|x| = R$. Then*

$$|\sigma(D_2^*, d|x|)g(x)| \leqslant C |g(x)|$$

where C is a constant independent of g.

Proof. Recall that $X_1, \ldots, X_d$ is our orthonormal basis for $\mathfrak{n}_1$ where $d = \dim(\mathfrak{n}_1)$. The entries of D_2^* are linear combinations of the X_i, $i = 1, \ldots, d$ with coefficients in $\mathbf{F}$. Let D_{ij} be the i, j entry, $1 \leqslant i \leqslant m_2$, $1 \leqslant j \leqslant m_3$. Then $D_{ij} = \sum_{k=1}^{d} C_{ij}^k X_k$. Thus, for $x \in S(R)$

$$\begin{aligned} |\sigma(D_2^*, d|x|)g(x)| &\leqslant C \sum_{i=1}^{m_2} \Big| \sum_{j=1}^{m_3} D_{ij}\big((|x|-R)g_j(x)\big) \Big| \\ &\leqslant C \sum_{i,j,k} | C_{ij}^k X_k\big((|x|-R)g_j(x)\big) | \\ &\leqslant C \sum_{jk} |(X_k|x|)g_j(x)| \qquad (\text{since } |x| = R) \\ &\leqslant C\big(\max_k (X_k|x|)\big) |g_j(x)|. \end{aligned}$$

We must show $X_k|x|$ is bounded. But $|x|$ is C^∞ away from the origin and homogeneous of degree 1. So $X_k|x|$ is homogeneous of degree 0. Thus, it is determined by its values on $\{|x| = 1\}$. It is C^∞ on this set and, therefore, bounded. This proves the lemma.

Since $h(R) \to h$ as $R \to \infty$ we have

LEMMA 3. *For* $\varepsilon > 0$, $\lim_{r\to\infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = h.$

We continue with the proof of our theorem. By the preceding lemma it suffices for us to prove that $\lim_{r\to\infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0$. But,

$$\frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \tag{14}$$

$$= \frac{1}{2\varepsilon} \int_{r-\varepsilon \leqslant |x| \leqslant r+\varepsilon} (D_1 D_1^* Kf, \sigma(D_2^*, d|x|) D_2 Kf)_x \, \| d \mid x \mid \| \, dx$$

because $d\mu_R dR = \| d \mid x \mid \| \, dx$. We claim that $\| d \mid x \mid \|$ is bounded. Let ω^{ij} be dual to X_{ij} where X_{ij} is our orthonormal basis. Then $d \mid x \mid = \sum_{i,j} (X_{ij}|x|)\omega^{ij}$. Since $\mid x \mid$ is homogeneous of degree 1 and X_{ij} is homogeneous of degree i we have $X_{ij} \mid x \mid$ is homogeneous of degree $1 - i$. Hence, for $\mid x \mid \geqslant 1$ each $X_{ij} \mid x \mid$ is bounded. So $\| d \mid x \mid \|$ is bounded.

By assumption, $f \in L^q(\mathbf{F}^{m_2})$, $q = Q/Q + 2$ and we know that $D_2 K$ is type 1. By Theorem 4(i) we know that $D_2 Kf \in L^2(\mathbf{F}^{m_2})$. Thus, by Lemma 2 $\mid \chi_r \sigma(D_2^*, d|x|) D_2 Kf \mid \leqslant C \mid \chi_r D_2 Kf \mid$ where χ_r is the characteristic function of $\{r - \varepsilon \leqslant \mid x \mid \leqslant r + \varepsilon\}$. We conclude that $\chi_r \sigma(D_2^*, d|x|) D_2 Kf \in L^2(\mathbf{F}^{m_2})$. By the Schwarz inequality and the fact that $\| d \mid x \mid \|$ is bounded, we get, from (14)

$$\frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR \leqslant \frac{c}{2\varepsilon} \| \chi_r D_1 D_1^* Kf \|_{L^2(\mathbf{F}^{m_2})} \| \chi_r D_2 Kf \|_{L^2(\mathbf{F}^{m_3})} .$$

As $r \to \infty$, both $\| \chi_r D_1 D_1^* Kf \|_{L^2(\mathbf{F}^{m_2})}$ and $\| \chi_r D_2 Kf \|_{L^2(\mathbf{F}^{m_3})}$ tend to 0. So $h = \lim_{r\to\infty} \frac{1}{2\varepsilon} \int_{r-\varepsilon}^{r+\varepsilon} h(R)dR = 0$. This proves the theorem.

This theorem together with Theorem 5 proves the Hodge decomposition. A similar argument gives the solution to the problem of finding g such that $D_1 g = f$ for a given f.

THEOREM 7. *Let* $f \in L^2(\mathbf{F}^{m_2}) \cap L^q(\mathbf{F}^{m_2})$ *with* $q = Q/Q + 2$. *Suppose* $D_2 f = 0$. *Then there exists* $g \in L^2(\mathbf{F}^{m_1})$ *such that* $D_1 g = f$.

Proof. We have $f = D_1 D_1^* Kf + D_2^* D_2 Kf$. It suffices to prove $(f, D_2^* D_2 Kf) = 0$ because this implies $D_2^* D_2 Kf = 0$ since

$$D_1 D_1^* Kf \perp D_2^* D_2 Kf .$$

We may set $g = D_1^* Kf$. Using the same notation as in the preceding theorem we have

$$(f, D_2^* D_2 Kf)_2 = \lim_{R \to \infty} (f, D_2^* D_2 Kf)_{D(R), 2}$$

$$= \lim_{R \to \infty} \big((D_2 f, D_2 Kf)_{D(R), 3} + (f, \sigma(D_2^*, d|x|) D_2 Kf)_{S(R), 2}\big)$$

$$= \lim_{R \to \infty} (f, \sigma(D_2^*, d|x|) D_2 Kf)_{S(R), 2} \quad (\text{since } D_2 f = 0).$$

The same argument as in Theorem 6 proves that the limit is zero. This proves the theorem.

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