PLANE CURVES IN FANCY BALLS

Autor(en): **Rudolph, Lee**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **31 (1985)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-54558

L'Enseignement Mathématique, t. 31 (1985), p. 81-84

PLANE CURVES IN FANCY BALLS

by Lee RUDOLPH [1])

One expects minimal surfaces (in the sense of differential geometry), and in particular complex analytic curves, to be subject to topological restrictions in the manner in which they may be embedded in a region of $\mathbf{R}^4$ (or $\mathbf{C}^2$). But caution is in order here: the ambient region, even if it is diffeomorphic to a ball, had better be subject to some geometrical hypotheses itself if it is to enforce good behavior on its minimal surfaces and complex curves. In fact we have the following result. Let D^4 be the standard, round, unit ball $\{(z, w) \in \mathbf{C}^2 : |z|^2 + |w|^2 \leqslant 1\}$.

THEOREM. *Let S be a 2-manifold-with-boundary smoothly embedded in D^4, subject only to the conditions that*

1. *S is orientable,*
2. *S has no closed components (i.e., is connected* rel ∂S*),*
3. *∂S is the transverse intersection of S and ∂D^4.*

Then there is a smooth embedding $i: D^4 \to \mathbf{C}^2$ such that $i(S)$ is a piece of complex curve, and a fortiori *a minimal surface in (flat) $\mathbf{R}^4 = \mathbf{C}^2$.*

Here, by "piece of complex curve" I mean a smooth oriented 2-manifold-with-boundary in $\mathbf{C}^2$ such that each of its tangent (real) 2-planes is in fact a complex line. (Then actually the interior of the 2-manifold-with-boundary is locally defined by the vanishing of complex-analytic functions.)

It is worth contrasting this with a quotation from a recent paper by Joel Hass [0]:

"COROLLARY 1.14. *The intersection of a 3-sphere in $\mathbf{C}^2$ with a 1-dimensional complex variety, which is non-singular and has genus zero inside the 3-sphere, is a ribbon knot. The intersection of such a variety with the corresponding 4-ball is a ribbon disk.*

---

[1]) Research partially supported by the Fonds National Suisse at the Mathematics Institute of the University of Geneva.

"The results ... extend to surfaces of higher genus than the disc ... with ... 'ribbon surface of genus g' replacing 'ribbon disc'."

See Remark 1, below, for a discussion of ribbons. Hass, of course, is speaking of *round* 3-spheres.

Proof. For arbitrarily large integers g, one can find a non-singular affine algebraic curve $V_f = \{(z, w) \in \mathbf{C}^2 : f(z, w) = 0\}$, $f(z, w) \in \mathbf{C}[z, w]$, which is diffeomorphic to a closed, connected smooth surface of genus g with a single point (at infinity) removed. (For instance, if $p(w) \in C[w]$ is sufficiently general, of odd degree $2k + 1 \geqslant 3$, the hyperelliptic curve V_f with $f(z, w) = z^2 + p(w)$ is such a curve, of genus $g = k(2k-1)$.) So on some such V_f, there is a subsurface S' diffeomorphic to S (using 1., 2., and the obvious compactness of S). The diffeomorphic surfaces S and S' are both given as subsets of $\mathbf{C}^2 = \mathbf{R}^4$. One easily sees that they are in fact ambient isotopic. (From 2., S has a handle decomposition with only 0- and 1-handles; the isotopy can be constructed handle by handle.) Let i be the restriction to D^4 of the final stage of the ambient isotopy. □

COROLLARY. *Let* $K \subset S^3 = \partial D^4$ *be an oriented smooth knot or link. Then there is an embedding* $i: D^4 \to \mathbf{C}^2$ *such that* $i(K)$ *bounds a piece of complex curve in* $i(D^4)$.

Remarks. (1) It is certainly not true that, even allowing preliminary isotopy of the surface in D^4, every smooth surface $S \subset D^4$ satisfying 1., 2., and 3. can be represented as a piece of complex curve in D^4 itself. A surface $S \subset D^4$ (satisfying 3.) is said to be *ribbon embedded* if $L \mid S$ is a Morse function without local maxima, where $L(z, w) = |z|^2 + |w|^2$; a surface is *ribbon* if it is isotopic to a ribbon embedded surface. Relative Morse theory shows that it is a genuine topological restriction on $S \subset D^4$ to be ribbon (for instance, the map $\pi_1(\partial D^4 - \partial S) \to \pi_1(D^4 - S)$, induced by inclusion, is surjective if S is ribbon, but need not be in general even when S satisfies 1., 2., and 3. — cf. [0], p. 102). But, as is well-known, Milnor's proof [2], that a Stein manifold of complex dimension k embedded in affine space has the homotopy type of a CW-complex of dimension no more than k, in fact proves more—namely, a statement about the nature of the embedding of the Stein manifold which, particularized to a piece of complex curve embedded in D^4 (satisfying 3.), says the curve is ribbon. (Hass draws the same conclusion from minimality of complex curves [0].)

More generally (but with the same proof) any piece of complex curve in any Stein domain D homeomorphic to D^4—e.g., a bidisk $D^2 \times D^2$—is a ribbon (relative to a plurisubharmonic exhaustion of D—L being one such for D^4).

(2) Presumably it is also not true that every isotopy type of oriented smooth knot or link in D^4 is represented by the (complete) boundary of a piece of complex curve in D^4 with its standard, round, geometry; cf. [3, 4, 5, 6, 7]. But at this writing the question remains open, to the author's knowledge.

(3) Let $D^2 \subset D^4$ be a disk (ribbon or otherwise), S a surface of genus 1, satisfying 3., with $\partial S = \partial D^2$. In the proof of the theorem for this case, one may take $f(z, w) = w^2 + z(z-1)(z-2)$, say, so that the completion (i.e., closure) of V_f in $\mathbf{CP}^2 \supset \mathbf{C}^2$ is a closed surface of genus 1, which represents three times the canonical generator in $H_2(\mathbf{CP}^2; \mathbf{Z})$.

Suppose $i(D^4) \frown V_f$ were precisely $S' = i(S)$. Then one could perform a surgery on the closure of V_f in $\mathbf{CP}^2$, replacing S' by $i(D^2)$. The resulting 2-sphere, smooth(ab)ly embedded in $\mathbf{CP}^2$, would still represent three times the generator—a situation long known to be impossible [1]. Thus $i(D^4) \frown V_f$ properly contains S'. Indeed, it must have some component which links $\partial S'$ (geometrically, not algebraically).

This example is also related to Stein theory. In fact, in a Stein domain, a piece of analytic complex curve (with boundary in the boundary of the domain) is the zero locus of some analytic function defined globally in the domain. In this example, the globally defined function $f \mid (\text{Int}\, i(D^4))$ is analytically irreducible, and its zero locus properly contains $i(\text{Int}\, S)$.

REFERENCES

[0] HASS, Joel. The geometry of the slice-ribbon problem. *Math. Proc. Camb. Phil. Soc. 94* (1983), 101-108.

[1] KERVAIRE, Michel A. and John W. MILNOR. On 2-spheres in 4-manifolds. *Proc. Nat. Acad. Sci. U.S.A. 47* (1961), 1651-1657.

[2] MILNOR, J. *Morse Theory.* Annals of Math. Studies 51 (1963).

[3] RUDOLPH, Lee. Algebraic Functions and Closed Braids. *Topology 22*, No. 2 (1983), 191-202.

[4] —— Constructions of quasipositive knots and links, I. In *Nœuds, tresses et singularités*, Monographie No. 31 de l'Enseignement Mathématique, Genève 1983, 233-245.

[5] —— Constructions of quasipositive knots and links, II. To appear.

[6] —— Some Knot Theory of Complex Plane Curves. In *Nœuds, tresses et singularités*, Monographie No. 31 de l'Enseignement Mathématique, Genève 1983, 99-122.

[7] —— Braided surfaces and Seifert ribbons for closed braids. *Comment. Math. Helvetici 58* (1983), 1-37.

(Reçu le 7 décembre 1983)

Lee Rudolph
Box 251
Adamsville
Rhode Island, 02801, USA