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A GEOMETRIC PROOF OF BIEBERBACH'S THEOREMS ON CRYSTALLOGRAPHIC GROUPS

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Pour Ariane et Georges

1. INTRODUCTION

In 1910 Bieberbach proved two celebrated theorems in response to Hilbert's 18th problem.

THEOREM I. *Every discrete group of isometries acting on the n-dimensional euclidean space* $\mathbf{R}^n$ *with compact fundamental domain contains* n *linearly independent translations.*

Groups which satisfy the hypothesis of Theorem I are called n-dimensional crystallographic groups.

THEOREM II. *For each fixed* n *there are only finitely many isomorphism classes of n-dimensional crystallographic groups.*

Bieberbach's original proof of Theorem I is based on Minkowski's Theorem on simultaneous rational approximation and is difficult to read. Shortly after it came out, Frobenius gave a more accessible proof which is based on an argument using the commutativity of unitary matrices. Frobenius's method has, in one form or another, become standard in the contemporary literature.

In this note we present a completely different approach to Theorem I which has its origins in Gromov's work on almost flat manifolds [5]. The new idea is to start with those rigid motions which have a very small rotation part (cf. § 2 for notation), and then proceed to show that, in fact, these motions are pure translations. The simplification which results from this approach is striking.

We also give a new proof of Theorem II which does not run via the usual algebraic characterization of a crystallographic group. Instead we shall use a method which is more in the spirit of Minkowski's geometry of numbers, from where Bieberbach's original arguments departed.

Since the material is standard, the exposition will be condensed. Yet some efforts have been made not to frustrate the reader by omitting details.

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2. Rigid motions

In this section we fix the notation and collect the necessary (and hopefully sufficient) rudiments from Linear Algebra.

We consider $\mathbf{R}^n$ as an euclidean vector space with the standard inner product. We use $|x|$ to denote the length of a vector $x \in \mathbf{R}^n$, and $\sphericalangle\,(x, y) \in [0, \pi]$ to denote the angle between two vectors. A rigid motion α (isometry of $\mathbf{R}^n$) will be expressed in the form

$$x \mapsto \alpha x = Ax + a \qquad (x \in \mathbf{R}^n)$$

where $A = \operatorname{rot} \alpha \in O(n)$ is an orthogonal map, called the *rotation part* of α, and $\alpha = \operatorname{trans} \alpha \in \mathbf{R}^n$ is a vector, called the *translation part.*

2.1. *The commutator* $[\alpha, \beta]$ of two rigid motions $x \mapsto \alpha x = Ax + a$ and $x \mapsto \beta x = Bx + b$ is defined as $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$. The following formulae are easily checked:

$$\operatorname{rot}[\alpha, \beta] = [A, B]$$

$$\operatorname{trans}[\alpha, \beta] = (A - id)b + (id - [A, B])b + A(id - B)A^{-1}a\,.$$

2.2. *Rotations.* For $A \in O(n)$ we define

$$m(A) = \max\left\{\, |Ax - x| / |x| \;\middle|\; x \in \mathbf{R}^n \backslash \{0\} \right\}.$$

Note that $|Ax - x| \leqslant m(A)|x|$ for $x \in \mathbf{R}^n$. *The set*

$$\text{(i)} \qquad E_A = \left\{\, x \in \mathbf{R}^n \;\middle|\; |Ax - x| = m(A)|x| \,\right\}$$

is a non trivial A-invariant subspace. This is immediately checked except perhaps for the part "$x, y \in E_A$ implies $x \pm y \in E_A$". This part follows from the equation

$$2m^2(A)(|x|^2 + |y|^2) = 2(|Ax - x|^2 + |Ay - y|^2) = |A(x+y) - (x+y)|^2$$
$$+ |A(x-y) - (x-y)|^2 \leqslant m^2(A)(|x+y|^2 + |x-y|^2) = 2m^2(A)(|x|^2 + |y|^2)$$