

# VII. The Main Theorem

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VII. THE MAIN THEOREM

We are now able to characterize the definite spaces whose topology is admissible (Def. 1). Refer to Definition 21 for "type condition".

**THEOREM 28** [20]. *Let  $\mathfrak{E}$  be a definite space in the sense of Definition 15. The following conditions are equivalent*

(i)  $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$  (cf. (1), (2), (3))

(ii)  $L_c(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E})$  ("the topology is admissible", Def. 1)

(iii)  $k$  is complete and  $\mathfrak{E}$  is the completion of a  $\aleph_0$ -dimensional space spanned by an orthogonal basis that satisfies the type condition.

*Proof.* (i)  $\Rightarrow$  (ii) holds trivially because  $L_s \subseteq L_{\perp\perp} \subseteq L_c$  by continuity of the form; (ii)  $\Rightarrow$  (iii) was carried out in Chapter V. Just as in [18] we can establish (iii)  $\Rightarrow$  (i). Let  $\mathfrak{U} \in L_c(\mathfrak{E})$ . Pick a maximal orthogonal family  $(v_i)_{i \in I}$  in  $\mathfrak{U}$  and extend it to a maximal orthogonal family  $(v_i)_{i \in J}$  in  $\mathfrak{E}$ . For  $x \in \mathfrak{E}$  we have by Lemma 27  $x = x' + x''$  where  $x' = \sum_I \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$  and  $x'' = \sum_J \langle x, v_i \rangle \langle v_i \rangle^{-1} v_i$ . Now  $x' \in \overline{\mathfrak{U}} = \mathfrak{U}$ . All that remains to be shown is  $x'' \in \mathfrak{U}^\perp$ . Now  $\mathfrak{U}^\perp$  is closed so it suffices to show that  $v_i \in \mathfrak{U}^\perp$  for all  $i \in J$ . To this end pick  $u \in \mathfrak{U}$  and decompose  $u = u' + u''$  (analogous to the decomposition of  $x$ ):  $u'' = u - u' \in \mathfrak{U} - \mathfrak{U} = \mathfrak{U}$ . Now  $\langle u'', v_i \rangle = 0$  for all  $i \in I$  so  $u'' = 0$  since  $(v_i)_{i \in I}$  is a maximal orthogonal family. From

$$0 = u'' = \sum_J \langle u, v_i \rangle \langle v_i \rangle^{-1} v_i$$

we obtain  $\langle u, v_i \rangle = 0$  ( $i \in J$ ). As  $u \in \mathfrak{U}$  was arbitrary this says that  $v_i \in \mathfrak{U}^\perp$  ( $i \in J$ ).  
 Q.E.D.

*Remark 29.* Let the definite space  $\mathfrak{E}$  be the completion of  $\mathfrak{F} = k(e_i)_{i \in \mathbb{N}}, (e_i)_{i \in \mathbb{N}}$  an orthogonal family (that does not necessarily satisfy the type condition). If  $k$  is complete then  $\mathfrak{E}$  is isometric to the  $k$ -space  $\widehat{\mathfrak{F}}$  of all sequences  $(\lambda_i)_{i \in \mathbb{N}} \in k^\mathbb{N}$  such that  $\lim_{\mathbb{N}} (2\phi\lambda_i + \phi\langle e_i \rangle) = \infty$  and equipped with the form  $\langle (\lambda_i), (\mu_i) \rangle = \sum_{\mathbb{N}} \lambda_i \mu_i \langle e_i \rangle$ . Indeed, the set  $\widehat{\mathfrak{F}}$  is a definite  $k$ -space and the map  $\Psi: (\lambda_i) \rightarrow \sum \lambda_i e_i$  is a well defined isometry  $\widehat{\mathfrak{F}} \rightarrow \Psi(\widehat{\mathfrak{F}}) \subset \mathfrak{E}$ . By the "infinite Pythagoras" we have  $\ker \Psi = 0$ ; on the other hand, Lemma 16 shows that  $\Psi$  is also surjective.

Thus all definite spaces that carry an admissible topology are (by Theorem 28) of the kind invented by Keller.

*Remark 30.* By Theorem 28 the isometry type of a definite space with admissible topology is characterized by the sequence  $(\langle e_i \rangle)_{i \in \mathbf{N}}$  where  $(e_i)_{i \in \mathbf{N}}$  is a maximal orthogonal family in  $\mathfrak{E}$ . Conversely, for each  $(\alpha_i) \in k^{\mathbf{N}}$  there is a definite space  $\mathfrak{E}$  with  $L_c(\mathfrak{E}) = L_s(\mathfrak{E})$  admitting a maximal orthogonal family  $(e_i)_{i \in \mathbf{N}}$  with  $\langle e_i \rangle = \alpha_i$  ( $i \in \mathbf{N}$ ) provided that

(A)  $\xi_i := \varphi \alpha_i \in \Gamma$  satisfies the (type-) condition expressed in (8)

(B) The form  $\langle \cdot, \cdot \rangle$  defined on  $\mathfrak{F} := k(e_i)_{i \in \mathbf{N}}$  by  $\langle e_i, e_j \rangle = 0$  ( $i \neq j$ ),  $\langle e_i \rangle = \alpha_i$  ( $i \in \mathbf{N}$ ) is definite.

These two conditions are implemented by many fields. In order to satisfy (A) one may, e.g. pick fields of generalized formal power series that are complete under a valuation  $\varphi$  with group  $\Gamma$  a prescribed Hahn product [30, p. 31] with sufficiently many factors not 2-divisible, e.g.  $\Gamma = \mathbf{Z}^{(\mathbf{N})}$  ordered antilexicographically. Let  $k$  be any field with (A) and  $t \in \Gamma/2\Gamma$ ; set  $\mathfrak{F}_t = \{\text{span } e_i \mid \varphi \alpha_i + 2\Gamma = t\}$ . By (A)  $\dim \mathfrak{F}_t < \infty$ ; furthermore

$$\mathfrak{F} = \bigoplus^\perp \{\mathfrak{F}_t \mid t \in \Gamma/2\Gamma\}.$$

In order to check whether the form  $\langle \cdot, \cdot \rangle$  satisfies the triangle inequality on  $\mathfrak{F}$  it suffices to verify said inequality on each  $\mathfrak{F}_t$ . A. Fässler has given a handy criterium for  $\langle \cdot, \cdot \rangle$  to be definite if Hahnproducts  $\Gamma$  are used, as indicated, to construct  $k$  with (A), [6, Lemma 15, 16].

## VIII. APPENDIX: EXTENDING THE MAIN THEOREM TO THE CLASS $\mathcal{E}$ OF NORM-TOPOLOGICAL SPACES

The arguments applied to the spaces in the class  $\mathcal{D}$  can be extended to a larger class  $\mathcal{E}$ . First we have (cf. Definition 15):

*Definition 31.* An infinite dimensional anisotropic quadratic space  $(\mathfrak{E}; \langle \cdot, \cdot \rangle)$  over a  $*$ -valued field  $(k, *, \varphi, \Gamma)$  is called norm-topological if the sets  $\mathcal{U}_\gamma := \{x \in \mathfrak{E} \mid \varphi \langle x \rangle > \gamma\}$  form a 0-neighbourhood basis of a vector space topology on  $\mathfrak{E}$ . Let  $\mathcal{E}$  be the class of all norm-topological spaces.

Definite spaces are norm-topological, obviously.

A proper subgroup  $\Delta$  of  $\Gamma$  is *convex* (or *isolated*) if “ $0 \leq x \leq y$  &  $y \in \Delta$ ” implies “ $x \in \Delta$ ”. If the subgroup  $\Delta \subset \Gamma$  is convex then the factor group  $\Gamma/\Delta$  is ordered by setting  $\gamma + \Delta \leq \delta + \Delta$  iff  $\gamma < \delta$  or  $\gamma - \delta \in \Delta$ ; furthermore,  $\varphi_\Delta: k \rightarrow \Gamma/\Delta \cup \{\infty\}$  defined by  $\varphi_\Delta(\alpha) = \varphi(\alpha) + \Delta$  is a valuation (a “coarser valuation”) which yields the same topology on  $k$  as  $\varphi$ .