

X. Clifford algebras of orthomodular spaces

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X. CLIFFORD ALGEBRAS OF ORTHOMODULAR SPACES

X.1. ASSUMPTIONS. In Chap. X k is a commutative field of characteristic not 2 and \langle , \rangle is a symmetric bilinear anisotropic form $\mathfrak{E} \times \mathfrak{E} \rightarrow k$ on the k -vector space \mathfrak{E} .

$C(\mathfrak{E})$ is the Clifford algebra of $(\mathfrak{E}; \langle , \rangle)$; it is a k -algebra that contains the space \mathfrak{E} as a set of ring generators which satisfy $x \cdot \eta + \eta \cdot x = 2\langle x, \eta \rangle$. For any pair of elements $c, d \in C(\mathfrak{E})$ there exists a finite orthogonal family e_0, \dots, e_n in \mathfrak{E} such that $c = \sum_I \alpha_I e_I, d = \sum_I \beta_I e_I$; here the summation index I runs over all subsets

$I = \{i_1 < \dots < i_r\}$ of $\{0, 1, \dots, n\}$ and $e_I := e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_r}$; the empty product e_\emptyset is the unit element in $C(\mathfrak{E})$.

There is a *canonical* symmetric bilinear form \langle , \rangle on $C(\mathfrak{E})$ which extends the given form on \mathfrak{E} ([5, 11, 22]). One has

$$(16) \quad \langle c, d \rangle = \sum_I \alpha_I \beta_I \prod_{i \in I} \langle e_i, e_i \rangle$$

From now on we shall assume that $(\mathfrak{E}; \langle , \rangle)$ is an infinite dimensional definite space.

X.2. CLIFFORD ALGEBRAS OF DEFINITE SPACES. In [6] Angela Fässler has proved that for certain definite orthomodular spaces \mathfrak{E} the algebra $C(\mathfrak{E})$ is a skew field; furthermore, the k -vector space $C(\mathfrak{E})$ equipped with the form (16) is a definite space whose completion $\tilde{C}(\mathfrak{E})$ is orthomodular again. Furthermore $\tilde{C}(\mathfrak{E})$ is a skew field, in fact, a $*$ -valued field with $*$ the extension to $\tilde{C}(\mathfrak{E})$ of the main antiautomorphism of the Clifford algebra $C(\mathfrak{E})$; the residue class field of $\tilde{C}(\mathfrak{E})$ is isomorphic to the residue class field of \mathfrak{E} .

In the following theorem we prove the main fact in a simplified and slightly more general setting.

THEOREM 37. Assume that in the definite space $(\mathfrak{E}; \langle , \rangle)$ each orthogonal family e_0, \dots, e_n has

$$(17) \quad \varphi\langle e_0 \rangle + \dots + \varphi\langle e_n \rangle \notin 2\Gamma$$

Then :

- (i) $C(\mathfrak{E})$ equipped with the form in (16) is a definite space,
- (ii) $C(\mathfrak{E})$ is a division ring,

(iii) The map $\tilde{\varphi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$ defined by $c \mapsto \varphi\langle c \rangle$ is a $*$ -valuation for $*$ the main antiautomorphism of $C(\mathfrak{E})$.

Proof. (i) It suffices to prove the triangle inequality (Lemma 14 (i)). Write $c = \sum \alpha_I e_I$, $d = \sum \beta_I e_I$ as in X.1. Then we have $\varphi\langle \alpha e_I \rangle \neq \varphi\langle \beta e_J \rangle$ for $I \neq J$ and $\alpha \neq 0 \neq \beta$. Hence

$$\varphi\langle c \rangle = \varphi\langle \sum \alpha_I e_I \rangle = \min_I \{\varphi\langle \alpha_I e_I \rangle\}$$

and similarly for $\varphi\langle d \rangle$. Therefore

$$\begin{aligned} \varphi\langle c+d \rangle &= \varphi\langle \sum (\alpha_I + \beta_I) e_I \rangle \geq \min \{2\varphi(\alpha_I + \beta_I) + \varphi\langle e_I \rangle\} \\ &\geq \min \{2\varphi\alpha_I + \varphi\langle e_I \rangle, 2\varphi\beta_I + \varphi\langle e_I \rangle\} = \min \{\varphi\langle c \rangle, \varphi\langle d \rangle\}. \end{aligned}$$

This proves (i). Next we show

$$(18) \quad \varphi\langle c \cdot d \rangle = \varphi\langle c \rangle + \varphi\langle d \rangle$$

Indeed, from

$$\langle e_I \cdot e_J \rangle = \langle \pm \langle e_{I \cap J} \rangle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_{I \cap J} \rangle^2 \langle e_{(I \cup J) \setminus (I \cap J)} \rangle = \langle e_I \rangle \cdot \langle e_J \rangle$$

we see that

$$\varphi\langle \alpha_I e_I \cdot e_J \rangle \leq \varphi\langle \alpha_I e_I \rangle \ \& \ \varphi\langle \beta_J e_J \cdot e_I \rangle \leq \varphi\langle \beta_J e_J \rangle$$

implies

$$\varphi\langle \alpha_I, \beta_J, e_I, e_J \rangle \leq \varphi\langle \alpha_I \beta_J e_I e_J \rangle.$$

We therefore pick $G, H \subseteq \{0, \dots, n\}$ such that for all $I \subset \{0, \dots, n\}$ we shall have

$$\varphi\langle \alpha_G e_G \rangle \leq \varphi\langle \alpha_I e_I \rangle, \varphi\langle \beta_H e_H \rangle \leq \varphi\langle \beta_I e_I \rangle.$$

It now follows that

$$\begin{aligned} \varphi\langle c \cdot d \rangle &= \varphi\langle (\sum \alpha_I e_I) \cdot \sum \beta_J e_J \rangle = \varphi\langle \sum \alpha_I \beta_J e_I e_J \rangle = \varphi\langle \alpha_G \beta_H e_G e_H \\ &\quad + \sum' \alpha_I \beta_J e_I e_J \rangle = \varphi\langle \alpha_G \beta_H e_G e_H \rangle = \varphi\langle c \rangle + \varphi\langle d \rangle. \end{aligned}$$

Thus (18) is established.

From (18) it follows that $C(\mathfrak{E})$ has no zero divisors, hence $C(\mathfrak{E})$ is a division ring (being an inductive limit of finite dimensional algebras). The map $\tilde{\varphi}: C(\mathfrak{E}) \rightarrow \Gamma \cup \{\infty\}$ as defined in (iii) of the Theorem is a $*$ -valuation, for $\tilde{\varphi}(c^*) = \tilde{\varphi}(c)$ is obvious and everything else has been established already.

COROLLARY 38. Assume that the definite space $(\mathfrak{E}; \langle , \rangle)$ is complete and that the system of types (Corollary 26) is linearly independent in $\Gamma/2\Gamma$ (considered as a \mathbf{Z}_2 -vector space) then the conclusions (i), (ii), (iii) of Theorem 37 hold.

$C(\mathfrak{E})$ in Theorem 37 is not complete (unless finite dimensional). Its quadratic form \langle , \rangle can be extended to the completion \tilde{C} . By using Theorem 28 one can see that this completion has $L_{\perp\perp}(\tilde{C}) = L_c(\tilde{C})$ if and only if E has $L_{\perp\perp}(E) = L_c(E)$.

XI. CONTINUOUS OPERATORS ARE NOT ALWAYS BOUNDED

XI.1. INTRODUCTION. Let \mathfrak{E} be an infinite dimensional definite space in the sense of Definition 15. A linear map (operator) $h: \mathfrak{E} \rightarrow \mathfrak{E}$ is called bounded iff there exists $\gamma \in \Gamma$ such that for all $x \in \mathfrak{E}$ we have $\varphi\langle hx \rangle \geq \gamma + \varphi\langle x \rangle$.

In [6] A. Fässler gave an explicit example of a continuous operator h on an orthomodular space \mathfrak{E} that is not bounded; she also proved a criterion for boundness which is very useful in the study of the algebra $\mathcal{B}(\mathfrak{E})$ of bounded operators $h: \mathfrak{E} \rightarrow \mathfrak{E}$ when \mathfrak{E} is an orthomodular definite space of a certain kind. We shall prove this criterion anew here as its original proof can be shortened considerably.

We shall consider definite spaces that satisfy

(19) $(\mathfrak{E}; \langle , \rangle)$ contains a maximal orthogonal family $(e_i)_{\mathbf{N}}$ such that the groups $\Theta(\varphi\langle e_i \rangle)$ are different.

By (14) we see that (19) is a property of \mathfrak{E} , not of $(e_i)_{\mathbf{N}}$; Keller's original example of an orthomodular space satisfies (19).

XI.2. FÄSSLER'S CRITERION. In this subsection let $(\mathfrak{E}; \langle , \rangle)$ be an infinite dimensional orthomodular space that has (19). Fix a maximal orthogonal family $(e_i)_{\mathbf{N}}$ that enjoys (19). If $f: \mathfrak{E} \rightarrow \mathfrak{E}$ is given, expand (Lemma 27)

$$(20) \quad f e_i = \sum_{j \in \mathbf{N}} \alpha_{ij} e_j \quad (i \in \mathbf{N})$$

THEOREM 39 ([6]). The linear map f is bounded iff it is continuous and satisfies