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XII. THE CLOSED GRAPH THEOREM

Let $\mathfrak{E}$, $\mathfrak{F}$ be definite spaces in the sense of Definition 15 over a field k whose valuation topology satisfies the 1. axiom of countability. For $f: \mathfrak{E} \to \mathfrak{F}$ a linear map set $\mathfrak{G}(f) := \{(\mathfrak{x}, \mathfrak{y}) \in \mathfrak{E} \oplus \mathfrak{F} \mid \mathfrak{y} = f(\mathfrak{x})\}$. Then [21] the "closed graph theorem" can be proved by classical methods (Baire category arguments):

(24) $$\mathfrak{G}(f) \text{ is closed} \Rightarrow f \text{ is continuous}.$$

There is the following algebraic analogue of statement (24):

(25) $$\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp} \Rightarrow f \text{ is } \perp\text{-continuous}$$

Here $\mathfrak{G}(f)^{\perp\perp}$ is taken in $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$ and, by definition, f is $\perp$-continuous iff f is continuous with respect to the topologies on $\mathfrak{E}$ and $\mathfrak{F}$ whose 0-neighbourhood filters are generated by the orthogonals of all *finite* dimensional subspaces of $\mathfrak{E}$ and $\mathfrak{F}$ respectively. For $\mathfrak{E}$ an orthomodular space implication (25) holds: $\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp}$ implies that $\mathfrak{G}(f)$ is closed since the form is continuous on $\mathfrak{E} \overset{\perp}{\oplus} \mathfrak{F}$; so f is continuous by (24). Further, if $\mathfrak{G} \subset \mathfrak{F}$ is the orthogonal of a finite dimensional subspace then $f^{-1}(\mathfrak{G})$ is closed, hence $f^{-1}(\mathfrak{G}) = (f^{-1}(\mathfrak{G}))^{\perp\perp}$ as $\mathfrak{E}$ is orthomodular. But $(f^{-1}(\mathfrak{G}))^{\perp}$ is finite dimensional, hence f is $\perp$-continuous.

In [31] nice examples of $f: \mathfrak{E} \to \mathfrak{F}$ are given which illustrate that (25) is in general violated.

XIII. A FEW OPEN PROBLEMS

All orthomodular spaces are meant to be infinite dimensional and different from the classical ones over **R**, **C**, **H**.

Problem 1. Are cardinalities of maximal orthogonal families in an orthomodular space always equal? The answer is "yes" for those in $\mathscr{E}$.

Problem 2. Give an example of an orthomodular space that contains an uncountable orthogonal family of non-zero vectors.

Problem 3. Does the implication

$$\mathfrak{A} + \mathfrak{B} = (\mathfrak{A}+\mathfrak{B})^{\perp\perp} \Rightarrow \mathfrak{A}^{\perp} + \mathfrak{B}^{\perp} = (\mathfrak{A}\cap\mathfrak{B})^{\perp}$$