

# XIII. A FEW OPEN PROBLEMS

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XII. THE CLOSED GRAPH THEOREM

Let  $\mathfrak{E}, \mathfrak{F}$  be definite spaces in the sense of Definition 15 over a field  $k$  whose valuation topology satisfies the 1. axiom of countability. For  $f: \mathfrak{E} \rightarrow \mathfrak{F}$  a linear map set  $\mathfrak{G}(f) := \{(x, \eta) \in \mathfrak{E} \oplus \mathfrak{F} \mid \eta = f(x)\}$ . Then [21] the “closed graph theorem” can be proved by classical methods (Baire category arguments):

$$(24) \quad \mathfrak{G}(f) \text{ is closed} \Rightarrow f \text{ is continuous .}$$

There is the following algebraic analogue of statement (24):

$$(25) \quad \mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp} \Rightarrow f \text{ is } \perp\text{-continuous}$$

Here  $\mathfrak{G}(f)^{\perp\perp}$  is taken in  $\mathfrak{E} \oplus^{\perp} \mathfrak{F}$  and, by definition,  $f$  is  $\perp$ -continuous iff  $f$  is continuous with respect to the topologies on  $\mathfrak{E}$  and  $\mathfrak{F}$  whose 0-neighbourhood filters are generated by the orthogonals of all *finite* dimensional subspaces of  $\mathfrak{E}$  and  $\mathfrak{F}$  respectively. For  $\mathfrak{E}$  an orthomodular space implication (25) holds:  $\mathfrak{G}(f) = \mathfrak{G}(f)^{\perp\perp}$  implies that  $\mathfrak{G}(f)$  is closed since the form is continuous on  $\mathfrak{E} \oplus^{\perp} \mathfrak{F}$ ; so  $f$  is continuous by (24). Further, if  $\mathfrak{G} \subset \mathfrak{F}$  is the orthogonal of a finite dimensional subspace then  $f^{-1}(\mathfrak{G})$  is closed, hence  $f^{-1}(\mathfrak{G}) = (f^{-1}(\mathfrak{G}))^{\perp\perp}$  as  $\mathfrak{E}$  is orthomodular. But  $(f^{-1}(\mathfrak{G}))^{\perp}$  is finite dimensional, hence  $f$  is  $\perp$ -continuous.

In [31] nice examples of  $f: \mathfrak{E} \rightarrow \mathfrak{F}$  are given which illustrate that (25) is in general violated.

XIII. A FEW OPEN PROBLEMS

All orthomodular spaces are meant to be infinite dimensional and different from the classical ones over  $\mathbf{R}, \mathbf{C}, \mathbf{H}$ .

*Problem 1.* Are cardinalities of maximal orthogonal families in an orthomodular space always equal? The answer is “yes” for those in  $\mathcal{E}$ .

*Problem 2.* Give an example of an orthomodular space that contains an uncountable orthogonal family of non-zero vectors.

*Problem 3.* Does the implication

$$\mathfrak{A} + \mathfrak{B} = (\mathfrak{A} + \mathfrak{B})^{\perp\perp} \Rightarrow \mathfrak{A}^{\perp} + \mathfrak{B}^{\perp} = (\mathfrak{A} \cap \mathfrak{B})^{\perp}$$

hold for all pairs of  $\perp$ -closed subspaces  $\mathfrak{A} = \mathfrak{A}^{\perp\perp}$ ,  $\mathfrak{B} = \mathfrak{B}^{\perp\perp}$  in an orthomodular space? The answer is “yes” for orthomodular spaces in  $\mathcal{E}$ . Cf. Remark 3 in [31]. More generally, are there other elementary lattice theoretic statements (in the sense of first order logic) that are valid in all  $L_{\perp\perp}(E)$  where  $\mathfrak{E}$  is orthomodular?

*Problem 4.* Are there spaces  $\mathfrak{E}$  in  $\mathcal{D}$ ,  $\mathcal{E}$  with  $L_s(\mathfrak{E}) = L_{\perp\perp}(\mathfrak{E}) \subsetneq L_c(\mathfrak{E})$ ?

*Problem 5.* An orthomodular space  $\mathfrak{E}$  in  $\mathcal{E}$  is *never* isometric to any of its proper subspaces  $\mathfrak{X}$ , although it does happen that  $\mathfrak{E}$  is similar to a proper subspace  $\mathfrak{X}$ . However, Keller’s space is not similar to any of its proper subspaces. Give an intrinsic description of the phenomenon. (See [21].)

*Problem 6.* Answer Keller’s question in § 3 of the introduction: When is  $\{A\}'$  commutative for selfadjoint  $A$  in the algebra  $\mathcal{B}(\mathfrak{H})$  of bounded operators  $\mathfrak{H} \rightarrow \mathfrak{H}$ ?

*Problem 7.* Let  $\mathfrak{E}$  be an orthomodular space in  $\mathcal{D}$  or  $\mathcal{E}$  such that the types of the members of a maximal orthogonal family are all different. Let  $\Lambda$  be the (countable) set of these types. For each choice of a family  $(\lambda_i)_{i \in \Lambda}$  of nonnegative real numbers with  $\sum_{\Lambda} \lambda_i = 1$  there is a probability distribution  $f: L_{\perp\perp}(\mathfrak{E}) \rightarrow [0, 1] \subset \mathbf{R}$  uniquely defined as follows: for  $\mathfrak{X} \in L_{\perp\perp}(\mathfrak{E})$  set  $f(\mathfrak{X}) := \sum_{i \in J} \lambda_i$  where the subset  $J \subseteq \Lambda$  consists of the types of the members of any orthogonal basis of  $\mathfrak{X}$ . We have  $f(\mathfrak{E}) = 1$ ,  $f(0) = 0$ ,  $f(\sum \mathfrak{X}_i) = \sum f(\mathfrak{X}_i)$  for any countable family  $\mathfrak{X}_0, \mathfrak{X}_1, \dots$  of mutually orthogonal ( $\perp$ -closed) subspaces. These are by no means all probability distributions on  $\mathfrak{E}$ . There is a host of other possibilities. Can one bring some order into this multitude?

*Problem 8.* Classify the definite spaces with admissible topology over fixed base field.

*Problem 9.* Study the orthogonal group of definite orthomodular spaces.