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1.1 *Definition*: The *third product* of any two elements  $\mathbf{A}$  and  $\mathbf{B}$  of  $\Pi_p \text{End } \wedge^p V$  is given by  $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B})) \in \Pi_p \text{End } \wedge^p V$ , where  $(\alpha\mathbf{A})(\alpha\mathbf{B})$  is the composition product of the shuffle products  $\alpha\mathbf{A} = e \cdot I \cdot \mathbf{A}$  and  $\alpha\mathbf{B} = e \cdot I \cdot \mathbf{B}$ .

Since the composition product is associative the third product is trivially associative. Furthermore, if  $I_0 \in \text{End } \wedge^0 V$  represents the unit element in  $\Pi_p \text{End } \wedge^p V$  with respect to the shuffle product one has

$$I_0 \times \mathbf{A} = \alpha^{-1}((\alpha I_0)(\alpha\mathbf{A})) = \alpha^{-1}((e \cdot I)(\alpha\mathbf{A})) = \alpha^{-1}(\mathbf{I}(\alpha\mathbf{A})) = \alpha^{-1}(\alpha\mathbf{A}) = \mathbf{A}$$

and similarly  $\mathbf{A} \times I_0 = \mathbf{A}$  for any  $\mathbf{A} \in \Pi_p \text{End } \wedge^p V$ ; that is,  $I_0$  is also the unit element of  $\Pi_p \text{End } \wedge^p V$  with respect to the third product. The rationale for introducing the third product appears in the next section.

## 2. THE TRACE

We now specialize the arbitrary  $R$ -module  $V$  of the preceding section.

2.1 *Definition*: A module  $V$  over a commutative ring  $R$  with unit is *traceable* of rank  $n > 0$  if and only if  $\text{End } \wedge^n V$  is a free  $R$ -module of rank one.

If  $\wedge^n V$  is itself free of rank one then  $V$  is clearly traceable of rank  $n$ . However,  $\text{End } \wedge^n V$  can be free of rank one with no such condition on  $\wedge^n V$ . For example, let  $X$  be any paracompact hausdorff space, let  $R$  be the ring  $C(X)$  of continuous real-valued functions on  $X$ , and let  $V$  be the  $C(X)$ -module of continuous sections of a real  $n$ -plane bundle  $\xi$  over  $X$ ; then  $V$  is traceable of rank  $n$ . However  $\wedge^n V$  is itself free of rank one if and only if  $\xi$  is orientable.

Flanders [1] showed for any module  $V$  over a commutative ring with unit that if  $\wedge^n V$  is free of rank one then  $\wedge^p V = 0$  for every  $p > n$ ; a similar argument shows that if  $V$  is traceable of rank  $n > 0$  then  $\text{End } \wedge^p V = 0$  for every  $p > n$ . Thus if  $V$  is traceable of rank  $n > 0$  there is no distinction between the direct product  $\Pi_p \text{End } \wedge^p V$  and the direct sum  $\amalg_p \text{End } \wedge^p V$ . Consequently the third product of Definition 1.1 can be regarded as a product in  $\amalg_p \text{End } \wedge^p V$  whenever  $V$  is traceable.

If  $V$  is traceable of rank  $n$  then every element of  $\text{End } \wedge^n V$  is scalar multiplication by a unique element of the commutative ground ring  $R$  with unit. For example, for any  $\mathbf{A} \in \amalg_p \text{End } \wedge^p V$  and each  $p = 0, \dots, n$  let

$(\alpha\mathbf{A})_p \in \text{End } \wedge^p V$  be the  $p^{\text{th}}$  component of  $\alpha\mathbf{A} \in \prod_p \text{End } \wedge^p V$ . Then  $(\alpha\mathbf{A})_n \in \text{End } \wedge^n V$  is scalar multiplication by a unique element of  $R$ .

2.2 *Definition*: If  $V$  is a traceable module of rank  $n > 0$  over a commutative ground ring  $R$  with unit, the *trace* of any  $\mathbf{A} \in \prod_p \text{End } \wedge^p V$  is the unique element  $\text{tr } \mathbf{A} \in R$  such that  $(\alpha\mathbf{A})_n = (\text{tr } \mathbf{A})I_n \in \text{End } \wedge^n V$ , for the identity endomorphism  $I_n \in \text{End } \wedge^n V$ .

For example, if  $A \in \text{End } V$  then  $(\alpha A)_n = A \cdot I_{n-1}$  for the identity endomorphism  $I_{n-1} \in \text{End } \wedge^{n-1} V$ . One easily verifies that if  $V$  is a free  $R$ -module of rank  $n$  then the classical trace of  $A$  is precisely that element  $\text{tr } A \in R$  such that  $A \cdot I_{n-1} = (\text{tr } A)I_n \in \text{End } \wedge^n V$ .

2.3 THEOREM. Let  $\prod_p \text{End } \wedge^p V$  be the endomorphism algebra generated by the endomorphisms of a traceable module  $V$ , multiplication being the third product; then the trace is an algebra homomorphism  $\prod_p \text{End } \wedge^p V \xrightarrow{\text{tr}} R$  over the ground ring  $R$ . Specifically, both  $\text{tr}(\mathbf{A} + \mathbf{B}) = \text{tr } \mathbf{A} + \text{tr } \mathbf{B}$  and  $\text{tr}(\mathbf{A} \times \mathbf{B}) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})$  for any elements  $\mathbf{A}$  and  $\mathbf{B}$  of  $\prod_p \text{End } \wedge^p V$ .

*Proof.* Additivity of the trace is trivial. To show that the trace also respects the third product suppose that  $V$  is traceable of rank  $n$ , and let  $(\alpha\mathbf{A})_p, (\alpha\mathbf{B})_p$  and  $\alpha(\mathbf{A} \times \mathbf{B})_p$  denote the components of  $\alpha\mathbf{A}$ ,  $\alpha\mathbf{B}$  and  $\alpha(\mathbf{A} \times \mathbf{B})$  in  $\text{End } \wedge^p V$  for each  $p = 0, \dots, n$ . By the definition  $\mathbf{A} \times \mathbf{B} = \alpha^{-1}((\alpha\mathbf{A})(\alpha\mathbf{B}))$  of the third product one has  $\alpha(\mathbf{A} \times \mathbf{B}) = (\alpha\mathbf{A})(\alpha\mathbf{B})$  for the composition product  $(\alpha\mathbf{A})(\alpha\mathbf{B})$ , that is,  $\prod_p \alpha(\mathbf{A} \times \mathbf{B})_p = \prod_p (\alpha\mathbf{A})_p (\alpha\mathbf{B})_p$ . In particular  $\alpha(\mathbf{A} \times \mathbf{B})_n = (\alpha\mathbf{A})_n (\alpha\mathbf{B})_n$  in the  $n^{\text{th}}$  component  $\text{End } \wedge^n V$ , so that

$$\text{tr}(\mathbf{A} \times \mathbf{B})I_n = ((\text{tr } \mathbf{A})I_n)((\text{tr } \mathbf{B})I_n) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})I_n$$

by definition of the trace; since  $\text{End } \wedge^n V$  is free on the single generator  $I_n$  this implies  $\text{tr}(\mathbf{A} \times \mathbf{B}) = (\text{tr } \mathbf{A})(\text{tr } \mathbf{B})$  as claimed.

### 3. PROPERTIES OF THE THIRD PRODUCT

We now establish several properties of the third product. Although these properties do not require the  $R$ -module  $V$  to be traceable, we shall later impose a condition on elements of the  $R$ -module  $\prod_r \text{End } \wedge^r V$  itself; the condition will automatically be satisfied in the applications.

Let  $V$  be any module over a commutative ring  $R$  with unit, and let  $\mathbf{A}$  and  $\mathbf{B}$  be elements of the direct product  $\prod_r \text{End } \wedge^r V$  whose only