Zeitschrift: L'Enseignement Mathématique

Band: 32 (1986)

Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: ON CONSECUTIVE VALUES OF THE LIOUVILLE FUNCTION

Kapitel: 4. Proof of the Theorem, conclusion

Autor: Hildebrand, Adolf

DOI: https://doi.org/10.5169/seals-55088

then

$$\lambda\left(\frac{N}{5} \pm 1\right) = \frac{\lambda(N \pm 5)}{\lambda(5)} = 1 = -\lambda(N) = \lambda\left(\frac{N}{5}\right),$$

and $N/5 = 4n/5 \in I_n$ is good. We may therefore suppose that at least one of values $\lambda(N+5)$ and $\lambda(N-5)$ equals 1.

For definiteness we shall assume $\lambda(N+5) = 1$; the other case is treated in exactly the same way.

If $\lambda(N+3) = 1$ or $\lambda(N+6) = 1$, then $N + 4 \in I_n$ or $N + 5 \in I_n$ is good. But in the remaining case

$$\lambda(N+3) = \lambda(N+6) = -1$$

we have

$$\lambda\left(\frac{N}{3}\right) = \lambda\left(\frac{N}{3} + 1\right) = \lambda\left(\frac{N}{3} + 2\right) = 1,$$

so that $(N+3)/3 \in I_n$ is good.

Thus (3) implies the existence of a good integer in the interval (4), as we had to show.

4. Proof of the Theorem, conclusion

So far we have proved that (1) has infinitely many solutions in the cases $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$ and $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$. But this obviously implies that for each of the triples $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, 1, -1), (-1, -1, 1), (1, -1, -1)$ and $(-1, 1, 1)$ there are also infinitely many solutions to (1). It remains therefore to consider the triples $(1, -1, 1)$ and $(-1, 1, -1)$. Since the arguments in both cases are the same (with $+1$ and -1 interchanged), we shall confine ourselves to the case $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$. Accordingly, we call $n \geqslant 2$ good, whenever

$$\lambda(n+1) = \lambda(n-1) = 1, \quad \lambda(n) = -1.$$

We have to show that there are infinitely many such n.

Suppose, to get a contradiction, that there are only finitely many good integers, all of them $\leqslant N_0$, say. Suppose further that

$$\lambda(n) = 1 (m_0 \leqslant n \leqslant n_0) \tag{5}$$

holds for some integers $n_0 > m_0 \geqslant 2N_0$. We shall show that then

$$\lambda(n) = 1 (m_i \leqslant n \leqslant n_i) \tag{6}$$

holds for all $i \geqslant 0$, where m_i and n_i are defined inductively by

$$m_{i+1} = \left[\frac{3m_i + 1}{2}\right], \quad n_{i+1} = \left[\frac{3n_i}{2}\right] (i \geqslant 0) . \tag{7}$$

This will easily lead to the desired contradiction.

By our assumption (5), (6) holds for $i = 0$. Assume now that (6) does not hold for all $i \geqslant 0$, and let $i \geqslant 0$ be minimal such that (6) holds for i and fails for $i + 1$. Thus, for some $n \in [m_{i+1}, n_{i+1}]$, which we shall fix, we have $\lambda(n) = -1$. Write

$$2n = 3n' + \theta(\theta \in \{0, 1, -1\}) . \tag{8}$$

From (7) we get

$$3m_i \leqslant 2m_{i+1} \leqslant 2n \leqslant 2n_{i+1} \leqslant 3n_i ,$$

so that

$$m_i \leqslant n' \leqslant n_i ,$$

and hence by (6) (which we assumed to hold for i)

$$\lambda(3n') = -\lambda(n') = -1 .$$

Since, by our assumption $\lambda(n) = -1$,

$$\lambda(2n) = -\lambda(n) = 1 ,$$

we cannot have $\theta = 0$ in (8). The arguments in the cases $\theta = \pm 1$ being identical, we shall henceforth assume that (8) holds with $\theta = 1$.

We must have

$$\lambda(2(n-1)) = \lambda(3n'-1) = -1 ,$$

since otherwise $3n'$ would be good and

$$3n' \geqslant 3m_i \geqslant 3m_0 > N_0 ,$$

in contradiction to our assumption. Also, since

$$m_i \leqslant n' + 1 = \frac{2}{3}(n+1) \leqslant \left[\frac{2}{3}(n_{i+1}+1)\right] = \left[\frac{2}{3}\left(\left[\frac{3n_i}{2}\right] + 1\right)\right] \leqslant n_i ,$$

we have by (6)

$$\lambda(2(n+1)) = \lambda(3(n'+1)) = -\lambda(n'+1) = -1 .$$

These two identities imply

$$\lambda(n \pm 1) = -\lambda(2(n \pm 1)) = 1,$$

and since $\lambda(n) = -1$, we conclude that $n(>N_0)$ is good and therefore arrive at a contradiction.

Thus (5) (with $n_0 > m_0 \geqslant 2N_0$) implies (6) for all $i \geqslant 0$. To derive from this the desired contradiction, we suppose first that (5) holds for some $n_0 > m_0 \geqslant 2N_0$ satisfying

$$n_0 - m_0 \geqslant 3. \tag{9}$$

In other words, we suppose (for the moment) that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. Putting $d_i = n_i - m_i$, we have, by the recursion formulae (7),

$$d_{i+1} \geqslant \frac{3}{2} d_i - 1 = \frac{3}{2} d_i \left(1 - \frac{2}{3d_i}\right) \qquad (i \geqslant 0).$$

Taking into account (9), we obtain by induction in turn

$$d_i \geqslant 3 \qquad (i \geqslant 0),$$

$$d_i \geqslant 3 \left(\frac{7}{6}\right)^i \qquad (i \geqslant 0),$$

and finally

$$d_i \geqslant \left(\frac{3}{2}\right)^i \prod_{j=0}^{i} \left(1 - \frac{2}{3d_j}\right) \geqslant C \left(\frac{3}{2}\right)^i \qquad (i \geqslant 0),$$

where

$$C = \prod_{j \geqslant 0} \left(1 - \frac{2}{9}\left(\frac{6}{7}\right)^j\right) > 0.$$

Since on the other hand by (7)

$$d_i \leqslant n_i \leqslant \left(\frac{3}{2}\right)^i n_0 \qquad (i \geqslant 0),$$

we see from (6), that there are arbitrary large values of x, such that $\lambda(n)$ is constant in the interval $[x(1-\varepsilon), x]$, where $\varepsilon = C/n_0$. But this is impossible since, for x sufficiently large, every such interval contains integers n and n' of the form

$$n = 4^a 9^b, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N}),$$

for which

$$\lambda(n) = 1, \quad \lambda(n') = -1.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some $m_0 > 2N_0$ with $n_0 = m_0 + 2$, and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1.$$

If m_0 is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1,$$

so that $(m_0 + 1)/2 > N_0$ is good, in contradiction to our assumption. But if m_0 is even, then defining m_1 and n_1 by (7), (6) holds for $i = 1$, and we have

$$m_1 \geqslant 2N_0, n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$, and the proof of the theorem is complete.

5. Concluding remarks

In the foregoing proof, the relevant property of the Liouville function was that $\lambda(n)$ is completely multiplicative and assumes only the values ± 1. Besides this, we used only the fact that $\lambda(2) = \lambda(3) = \lambda(5) = -1$ and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1, \quad \lambda(29) = \lambda(31) = -1.$$

The proof, as it stands, works for any completely multiplicative function $f(n) = \pm 1$ with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.