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Kapitel: 5. CONCLUDING REMARKS

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$$n = 4^a 9^b\,, \quad n' = 2 \cdot 4^c 9^d (a, b, c, d \in \mathbf{N})\,,$$

for which

$$\lambda(n) = 1\,, \quad \lambda(n') = -1\,.$$

We therefore have obtained the desired contradiction under the assumption that there exist four consecutive integers $n \geqslant 2N_0$, for which $\lambda(n) = 1$. By the part of the theorem already proved, there exist at least three such integers. Therefore (5) holds for some $m_0 > 2N_0$ with $n_0 = m_0 + 2$, and we may now assume that

$$\lambda(m_0 - 1) = \lambda(m_0 + 3) = -1\,.$$

If m_0 is odd, then this implies

$$\lambda\left(\frac{m_0 - 1}{2}\right) = \lambda\left(\frac{m_0 + 3}{2}\right) = 1\,, \quad \lambda\left(\frac{m_0 + 1}{2}\right) = -1\,,$$

so that $(m_0 + 1)/2 > N_0$ is good, in contradiction to our assumption. But if m_0 is even, then defining m_1 and n_1 by (7), (6) holds for $i = 1$, and we have

$$m_1 \geqslant 2N_0\,, n_1 - m_1 = \frac{3(m_0 + 2)}{2} - \frac{3m_0}{2} = 3\,.$$

Thus we are back in the case already treated.

By contradiction, we therefore conclude that (1) has infinitely many solutions for $(\varepsilon_1, \varepsilon_2, \varepsilon_3) = (1, -1, 1)$, and the proof of the theorem is complete.

5. Concluding remarks

In the foregoing proof, the relevant property of the Liouville function was that $\lambda(n)$ is completely multiplicative and assumes only the values ± 1. Besides this, we used only the fact that $\lambda(2) = \lambda(3) = \lambda(5) = -1$ and (in the proof of the lemma)

$$\lambda(14) = \lambda(16) = 1\,, \quad \lambda(29) = \lambda(31) = -1\,.$$

The proof, as it stands, works for any completely multiplicative function $f(n) = \pm 1$ with these properties. By suitably modifying the proof, it is possible to cover other classes of multiplicative functions as well.

It would be interesting to determine those completely multiplicative functions $f(n) = \pm 1$, for which the analogue of the theorem does not hold. Schur [3] proved that if $f \not\equiv f_{\pm}$, where

$$f_{\pm}(n) = \begin{cases} (\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 1 \bmod 3, \\ -(\pm 1)^k & \text{if} \quad n = 3^k m, m \equiv 2 \bmod 3, \end{cases}$$

then there exists at least one $n \geqslant 1$, such that

$$f(n) = f(n+1) = f(n+2) = 1 .$$

It is likely that under the same hypotheses there are infinitely many such n. Using arguments similar to those in section 3, one can prove this assertion under the additional hypotheses $f(2) = 1$ and $f(3) = -1$, but the general case seems to be more complicated.

A very plausible conjecture is that the integers n, for which (1) holds, have positive density. In the case $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 1$, this would follow from an analogous strengthening of the lemma by requiring (2) to hold on a set of positive density. Whereas a very simple argument shows that the equations $\lambda(n) = \lambda(n+1)$ and $\lambda(n+1) = \lambda(n-1)$ hold on a set of positive (lower) density (cf. [2]), this argument seems to break down, if n is required to lie in a prescribed residue class, and so far we have not been able to overcome this difficulty.

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