

## §4. Variants of E (See [SKK], [Bj], [S])

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### § 3. THE ALGEBRAIC PROPERTIES OF $\mathcal{E}$ (See [SKK], [Bj])

3.1. In the preceding section, we introduced the notion of micro-differential operators. The ring  $\mathcal{E}$  of micro-differential operators has nice algebraic properties similar to those of the ring of holomorphic functions.

Let us recall some definitions of finiteness properties.

*Definition 3.1.1.* Let  $\mathcal{A}$  be a sheaf of rings on a topological space  $S$ .

- (1) An  $\mathcal{A}$ -module  $\mathcal{M}$  is called *of finite type* (resp. *of finite presentation*) if for any point  $x \in X$  there exists a neighborhood  $U$  and an exact sequence  $0 \leftarrow \mathcal{M}|_U \leftarrow \mathcal{A}^p|_U$  (resp.  $0 \leftarrow \mathcal{M}|_U \leftarrow \mathcal{A}^p|_U \leftarrow \mathcal{A}^q|_U$ ).
- (2)  $\mathcal{M}$  is called *pseudo-coherent*, if any submodule of finite type defined on an open subset is of finite presentation. If  $\mathcal{M}$  is pseudo-coherent and of finite type, then  $\mathcal{M}$  is called *coherent*.
- (3)  $\mathcal{M}$  is called *Noetherian* if  $\mathcal{M}$  satisfies the following properties:
  - (a)  $\mathcal{M}$  is coherent.
  - (b) For any  $x \in X$ ,  $\mathcal{M}_x$  is a Noetherian  $\mathcal{A}_x$ -module (i.e. any increasing sequence of  $\mathcal{A}_x$ -submodules is stationary).
  - (c) For any open subset  $U$ , any increasing sequence of coherent  $(\mathcal{A}|_U)$ -submodules of  $\mathcal{M}|_U$  is locally stationary.

As for the sheaf of holomorphic functions, we have

**THEOREM 3.1.1** ([SKK] Chap. II, Thm. 3.4.1, Prop. 3.2.7). *Let  $\overset{\circ}{T^*X}$  denote the complement of the zero section in  $T^*X$ .*

- (1)  $\mathcal{E}_X$  and  $\mathcal{E}_X(0)$  are Noetherian rings on  $T^*X$ .
- (2)  $\mathcal{E}_X$  is flat over  $\pi^{-1}\mathcal{D}_X$ .
- (3)  $\mathcal{E}_X(\lambda)|_{\overset{\circ}{T^*X}}$  is a Noetherian  $\mathcal{E}_X(0)|_{\overset{\circ}{T^*X}}$ -module.
- (4) For  $p \in T^*X$ ,  $\mathcal{E}_X(0)_p$  is a local ring with the residual field  $\mathbb{C}$ .
- (5) A coherent  $\mathcal{E}_X$ -module is pseudo-coherent over  $\mathcal{E}_X(0)$ .

### § 4. VARIANTS OF $\mathcal{E}$ (See [SKK], [Bj], [S])

4.1. We have defined the sheaf of rings  $\mathcal{E}$ . However we can introduce other sheaves of rings, similar to  $\mathcal{E}$ , which makes the theory transparent.

4.2. The sheaf  $\hat{\mathcal{E}} = \varprojlim_{m \in \mathbf{N}} \mathcal{E}/\mathcal{E}(-m)$  is called the sheaf of *formal micro-differential operators*. This is nothing but the sheaf similar to  $\mathcal{E}$ , obtained by dropping the growth condition (2.2.1).

4.3. We can define the sheaf  $\mathcal{E}^\infty$  of micro-differential operators of infinite order ([SKK]). For an open  $\Omega \subset \mathbf{C}^n$ , we set

$$\Gamma(\Omega; \mathcal{E}^\infty) = \{(p_j)_{j \in \mathbf{Z}}; p_j \in \Gamma(\Omega; \mathcal{O}_{T^*X}(j))\}$$

satisfying the following conditions (4.3.1) and (4.3.2).

(4.3.1) For any compact set  $K \subset \Omega$ , there is a  $C_K > 0$  such that  $\sup_K |p_j| \leq C_K^{-j} (-j)!$  for  $j < 0$ .

(4.3.2) For any compact set  $K \subset \Omega$  and any  $\varepsilon > 0$ , there exists a  $C_{K, \varepsilon} > 0$  such that

$$\sup_K |p_j| \leq C_{K, \varepsilon} \frac{\varepsilon^j}{j!} \quad \text{for } j \geq 1.$$

4.4. We can also define the sheaf  $\mathcal{E}^{\mathbf{R}}$  on  $T^*X$  by  $\mathcal{H}^n(\mu_\Delta(\mathcal{O}_{X \times X}^{(0, n)}))$ . (See [KS] Chap. II, [SKK]). Here  $n = \dim X$ ,  $\mathcal{O}_{X \times X}^{(0, n)}$  is the sheaf of holomorphic forms on  $X \times X$  which are  $n$ -forms with respect to the second variable, and  $\mu_\Delta$  is the micro-localization with respect to the diagonal set of  $X \times X$  (See [SKK] Chap. II for the details).

4.5. We have  $\mathcal{E}_X \subset \mathcal{E}_X^\infty \subset \mathcal{E}_X^{\mathbf{R}}$ ,  $\mathcal{E}_X \subset \hat{\mathcal{E}}_X$ . Moreover,  $\mathcal{E}_X^\infty$ ,  $\mathcal{E}_X^{\mathbf{R}}$  and  $\hat{\mathcal{E}}_X$  are faithfully flat over  $\mathcal{E}_X$ . The sheaf  $\hat{\mathcal{E}}_X$  is Noetherian. The sheaf  $\mathcal{E}_X^{\mathbf{R}}$  contains  $\mathcal{E}_X(\lambda)$ 's compatible with the multiplication.

4.6. If we denote by  $\gamma$  the projection map  $T^*X \rightarrow T^*X/\mathbf{C}^*$ , then  $R^j \gamma_* \mathcal{E}^{\mathbf{R}} = 0$  for  $j \neq 0$  and  $\mathcal{E}^\infty = \gamma^{-1} \gamma_* \mathcal{E}^{\mathbf{R}}$ .

4.7. In [SKK],  $\mathcal{E}$ ,  $\hat{\mathcal{E}}$ , and  $\mathcal{E}^\infty$  are denoted by  $\mathcal{P}^f$ ,  $\hat{\mathcal{P}}$  and  $\mathcal{P}$ .

4.8. To explain the differences between  $\mathcal{E}$ ,  $\mathcal{E}^\infty$ ,  $\mathcal{E}^{\mathbf{R}}$  and  $\hat{\mathcal{E}}$ , we shall take the following example. Let  $X$  be a complex manifold and  $Y$  a hypersurface of  $X$ . We shall take local coordinates  $(x_1, \dots, x_n)$  of  $X$  such that  $Y$  is given by  $x_1 = 0$ . The  $\mathcal{D}_X$ -module  $\mathcal{D}_X/\mathcal{D}_X x_1 + \sum_{j>1} \mathcal{D}_X \partial_j$  is denoted by  $\mathcal{B}_{Y|X}$ . Set

$$\begin{aligned} \mathcal{C}_{Y|X} &= \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}_X} \mathcal{B}_{Y|X}, \quad \widehat{\mathcal{C}}_{Y|X} = \widehat{\mathcal{E}}_X \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X}, \\ \mathcal{C}_{Y|X}^\infty &= \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X} \quad \text{and} \quad \mathcal{C}_{Y|X}^{\mathbf{R}} = \mathcal{E}_X^{\mathbf{R}} \otimes_{\mathcal{E}_X} \mathcal{C}_{Y|X}. \end{aligned}$$

Then we have, setting  $p = (0, dx_1)$ ,  $x_0 = 0$

$$\begin{aligned} \mathcal{C}_{Y|X,p} &= \{a + b \log x_1; a \in \mathcal{O}_{X,x_0}[1/x_1], b \in \mathcal{O}_{X,x_0}\} / \mathcal{O}_{X,x_0} \\ &\cong (\mathcal{O}_{X,x_0}[1/x_1] / \mathcal{O}_{X,x_0}) \oplus \mathcal{O}_{X,x_0} \\ \widehat{\mathcal{C}}_{Y|X,p} &= \{a + b \log x_1; a \in \mathcal{O}_{X,x_0}[1/x_1], b \in \widehat{\mathcal{O}}_{X|Y,x_0}\} / \mathcal{O}_{X,x_0} \\ &\cong (\mathcal{O}_{X,x_0}[1/x_1] / \mathcal{O}_{X,x_0}) \oplus \widehat{\mathcal{O}}_{X|Y,x_0}. \end{aligned}$$

Here  $\widehat{\mathcal{O}}_{X|Y} = \varprojlim \mathcal{O}_X / x_1^m \mathcal{O}_X$  is the sheaf of formal power series in the  $x_1$ -direction.

$$\mathcal{C}_{Y|X,p}^\infty = \{a + b \log x_1; a \in (j_* j^{-1} \mathcal{O}_X)_{x_0}, b \in \mathcal{O}_{X,x_0}\} / \mathcal{O}_{X,x_0}$$

where  $j$  is the open embedding  $X \setminus Y \hookrightarrow X$ .

$$\mathcal{C}_{Y|X,p}^{\mathbf{R}} = \varinjlim_U \mathcal{O}(U) / \mathcal{O}_{X,x_0}.$$

Here  $U$  ranges over the set of open subsets of the form

$$\{x \in X; |x| < \varepsilon, \operatorname{Re} x_1 < \varepsilon \operatorname{Im} x_1\}.$$

4.8. If we use  $\mathcal{E}_X^\infty$ , the structure of  $\mathcal{E}$ -modules becomes simpler. We just mention two theorems in this direction.

**THEOREM 4.8.1** ([KK] Thm. 5.2.1). *Let  $\mathcal{M}$  be a holonomic  $\mathcal{E}_X$ -module. Then there exists a (unique) regular holonomic  $\mathcal{E}_X$ -module  $\mathcal{M}_{\text{reg}}$  such that*

$$\mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M} \cong \mathcal{E}_X^\infty \otimes_{\mathcal{E}_X} \mathcal{M}_{\text{reg}}.$$

**THEOREM 4.8.2** ([SKK] Chap. II, Thm. 5.3.1). *Let  $X$  and  $Y$  be complex manifolds and let  $T_Y^*Y$  be the zero section of  $T^*Y$ . If  $\mathcal{M}$  is an  $\mathcal{E}_{X \times Y}$ -module whose support is contained in  $T^*X \times T_Y^*Y$ , then there exists a (locally) coherent  $\mathcal{E}_X$ -module  $\mathcal{L}$  such that*

$$\mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} \mathcal{M} \cong \mathcal{E}_{X \times Y}^\infty \otimes_{\mathcal{E}_{X \times Y}} (\mathcal{L} \widehat{\otimes} \mathcal{O}_Y).$$

Here  $\widehat{\otimes}$  denotes the exterior tensor product. (See § 8).