§4. Variants of E (See [SKK], [Bj], [S])

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§ 3. The Algebraic Properties of $\mathscr{E}$ (See [SKK], [Bj])

3.1. In the preceding section, we introduced the notion of micro-differential operators. The ring $\mathscr{E}$ of micro-differential operators has nice algebraic properties similar to those of the ring of holomorphic functions.

Let us recall some definitions of finiteness properties.

Definition 3.1.1. Let $\mathscr{A}$ be a sheaf of rings on a topological space S.

(1) An $\mathscr{A}$-module $\mathscr{M}$ is called *of finite type* (resp. *of finite presentation*) if for any point $x \in X$ there exists a neighborhood U and an exact sequence $0 \leftarrow \mathscr{M}|_U \leftarrow \mathscr{A}^p|_U$ (resp. $0 \leftarrow \mathscr{M}|_U \leftarrow \mathscr{A}^p|_U \leftarrow \mathscr{A}^q|_U$).

(2) $\mathscr{M}$ is called *pseudo-coherent*, if any submodule of finite type defined on an open subset is of finite presentation. If $\mathscr{M}$ is pseudo-coherent and of finite type, then $\mathscr{M}$ is called *coherent*.

(3) $\mathscr{M}$ is called *Noetherian* if $\mathscr{M}$ satisfies the following properties:

 (a) $\mathscr{M}$ is coherent.

 (b) For any $x \in X$, $\mathscr{M}_x$ is a Noetherian $\mathscr{A}_x$-module (i.e. any increasing sequence of $\mathscr{A}_x$-submodules is stationary).

 (c) For any open subset U, any increasing sequence of coherent $(\mathscr{A}|_U)$-submodules of $\mathscr{M}|_U$ is locally stationary.

As for the sheaf of holomorphic functions, we have

Theorem 3.1.1 ([SKK] Chap. II, Thm. 3.4.1, Prop. 3.2.7). *Let* $\mathring{T}^*X$ *denote the complement of the zero section in* T^*X.

(1) $\mathscr{E}_X$ *and* $\mathscr{E}_X(0)$ *are Noetherian rings on* T^*X.

(2) $\mathscr{E}_X$ *is flat over* $\pi^{-1}\mathscr{D}_X$.

(3) $\mathscr{E}_X(\lambda)|_{\mathring{T}^*X}$ *is a Noetherian* $\mathscr{E}_X(0)|_{\mathring{T}^*X}$*-module.*

(4) *For* $p \in T^*X$, $\mathscr{E}_X(0)_p$ *is a local ring with the residual field* **C**.

(5) *A coherent* $\mathscr{E}_X$*-module is pseudo-coherent over* $\mathscr{E}_X(0)$.

§ 4. Variants of $\mathscr{E}$ (See [SKK], [Bj], [S])

4.1. We have defined the sheaf of rings $\mathscr{E}$. However we can introduce other sheaves of rings, similar to $\mathscr{E}$, which makes the theory transparent.

4.2. The sheaf $\hat{\mathscr{E}} = \varprojlim_{m \in \mathbf{N}} \mathscr{E}/\mathscr{E}(-m)$ is called the sheaf of *formal micro-differential operators.* This is nothing but the sheaf similar to $\mathscr{E}$, obtained by dropping the growth condition (2.2.1).

4.3. We can define the sheaf $\mathscr{E}^{\infty}$ of micro-differential operators of infinite order ([SKK]). For an open $\Omega \subset \mathbf{C}^n$, we set

$$\Gamma(\Omega; \mathscr{E}^{\infty}) = \{(p_j)_{j \in \mathbf{Z}}; \quad p_j \in \Gamma(\Omega; \mathscr{O}_{T*X}(j))$$

satisfying the following conditions (4.3.1) and (4.3.2)$\}$.

(4.3.1) For any compact set $K \subset \Omega$, there is a $C_K > 0$ such that $\sup_K |p_j| \leqslant C_K^{-j}(-j)!$ for $j < 0$.

(4.3.2) For any compact set $K \subset \Omega$ and any $\varepsilon > 0$, there exists a $C_{K,\varepsilon} > 0$ such that

$$\sup_K |p_j| \leqslant C_{K,\varepsilon} \frac{\varepsilon^j}{j!} \quad \text{for} \quad j \geqslant 1.$$

4.4. We can also define the sheaf $\mathscr{E}^{\mathbf{R}}$ on T^*X by $\mathscr{H}^n(\mu_\Delta(\mathscr{O}_{X \times X}^{(0,n)}))$. (See [KS] Chap. II, [SKK]). Here $n = \dim X$, $\mathscr{O}_{X \times X}^{(0,n)}$ is the sheaf of holomorphic forms on $X \times X$ which are n-forms with respect to the second variable, and μ_Δ is the micro-localization with respect to the diagonal set of $X \times X$ (See [SKK] Chap. II for the details).

4.5. We have $\mathscr{E}_X \subset \mathscr{E}_X^{\infty} \subset \mathscr{E}_X^{\mathbf{R}}$, $\mathscr{E}_X \subset \hat{\mathscr{E}}_X$. Moreover, $\mathscr{E}_X^{\infty}$, $\mathscr{E}_X^{\mathbf{R}}$ and $\hat{\mathscr{E}}_X$ are faithfully flat over $\mathscr{E}_X$. The sheaf $\hat{\mathscr{E}}_X$ is Noetherian. The sheaf $\mathscr{E}_X^{\mathbf{R}}$ contains $\mathscr{E}_X(\lambda)$'s compatible with the multiplication.

4.6. If we denote by γ the projection map $T^*X \to T^*X/\mathbf{C}^*$, then $R^j\gamma_*\mathscr{E}^{\mathbf{R}} = 0$ for $j \neq 0$ and $\mathscr{E}^{\infty} = \gamma^{-1}\gamma_*\mathscr{E}^{\mathbf{R}}$.

4.7. In [SKK], $\mathscr{E}$, $\hat{\mathscr{E}}$, and $\mathscr{E}^{\infty}$ are denoted by $\mathscr{P}^f$, $\hat{\mathscr{P}}$ and $\mathscr{P}$.

4.8. To explain the differences between $\mathscr{E}$, $\mathscr{E}^{\infty}$, $\mathscr{E}^{\mathbf{R}}$ and $\hat{\mathscr{E}}$, we shall take the following example. Let X be a complex manifold and Y a hypersurface of X. We shall take local coordinates $(x_1, \ldots, x_n)$ of X such that Y is given by $x_1 = 0$. The $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X x_1 + \sum_{j>1} \mathscr{D}_X \partial_j$ is denoted by $\mathscr{B}_{Y|X}$. Set

$$\mathscr{C}_{Y|X} = \mathscr{E}_X \underset{\pi^{-1}\mathscr{D}_X}{\otimes} \mathscr{B}_{Y|X}, \hat{\mathscr{C}}_{Y|X} = \hat{\mathscr{E}}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X},$$

$$\mathscr{C}^{\infty}_{Y|X} = \mathscr{E}^{\infty}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X} \quad \text{and} \quad \mathscr{C}^{\mathbf{R}}_{Y|X} = \mathscr{E}^{\mathbf{R}}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{C}_{Y|X}.$$

Then we have, setting $p = (0, dx_1)$, $x_0 = 0$

$$\mathscr{C}_{Y|X,p} = \{a + b \log x_1; a \in \mathscr{O}_{X,x_0}[1/x_1], b \in \mathscr{O}_{X,x_0}\}/\mathscr{O}_{X,x_0}$$

$$\cong (\mathscr{O}_{X,x_0}[1/x_1]/\mathscr{O}_{X,x_0}) \oplus \mathscr{O}_{X,x_0}$$

$$\hat{\mathscr{C}}_{Y|X,p} = \{a + b \log x_1; a \in \mathscr{O}_{X,x_0}[1/x_1], b \in \hat{\mathscr{O}}_{X|Y,x_0}\}/\mathscr{O}_{X,x_0}$$

$$\cong (\mathscr{O}_{X,x_0}[1/x_1]/\mathscr{O}_{X,x_0}) \oplus \hat{\mathscr{O}}_{X|Y,x_0}.$$

Here $\hat{\mathscr{O}}_{X|Y} = \varprojlim \mathscr{O}_X/x_1^m\mathscr{O}_X$ is the sheaf of formal power series in the x_1-direction.

$$\mathscr{C}^{\infty}_{Y|X,p} = \{a + b \log x_1; a \in (j_*j^{-1}\mathscr{O}_X)_{x_0}, \quad b \in \mathscr{O}_{X,x_0}\}/\mathscr{O}_{X,x_0}$$

where j is the open embedding $X\backslash Y \hookrightarrow X$.

$$\mathscr{C}^{\mathbf{R}}_{Y|X,p} = \varinjlim_{U} \mathscr{O}(U)/\mathscr{O}_{X,x_0}.$$

Here U ranges over the set of open subsets of the form

$$\{x \in X; |x| < \varepsilon, \operatorname{Re} x_1 < \varepsilon \operatorname{Im} x_1\}.$$

4.8. If we use $\mathscr{E}^{\infty}_X$, the structure of $\mathscr{E}$-modules becomes simpler. We just mention two theorems in this direction.

THEOREM 4.8.1 ([KK] Thm. 5.2.1). *Let $\mathscr{M}$ be a holonomic $\mathscr{E}_X$-module. Then there exists a (unique) regular holonomic $\mathscr{E}_X$-module $\mathscr{M}_{\text{reg}}$ such that*

$$\mathscr{E}^{\infty} \underset{\mathscr{E}_X}{\otimes} \mathscr{M} \cong \mathscr{E}^{\infty} \underset{\mathscr{E}_X}{\otimes} \mathscr{M}_{\text{reg}}.$$

THEOREM 4.8.2 ([SKK] Chap. II, Thm. 5.3.1). *Let X and Y be complex manifolds and let T^*_YY be the zero section of T^*Y. If $\mathscr{M}$ is an $\mathscr{E}_{X\times Y}$-module whose support is contained in $T^*X \times T^*_YY$, then there exists a (locally) coherent $\mathscr{E}_X$-module $\mathscr{L}$ such that*

$$\mathscr{E}^{\infty}_{X\times Y} \underset{\mathscr{E}_{X\times Y}}{\otimes} \mathscr{M} \cong \mathscr{E}^{\infty}_{X\times Y} \underset{\mathscr{E}_{X\times Y}}{\otimes} (\mathscr{L}\hat{\otimes}\mathscr{O}_Y).$$

Here $\hat{\otimes}$ denotes the exterior tensor product. (See § 8).