

Zeitschrift: L'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INTRODUCTION TO MICROLOCAL ANALYSIS
Kapitel: §5. The Vanishing Cycle Sheaf
Autor: Kashiwara, Masaki
DOI: <https://doi.org/10.5169/seals-55089>

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§ 5. THE VANISHING CYCLE SHEAF

5.1. Let M be a real manifold and $f: M \rightarrow \mathbf{R}$ a continuous map. For a sheaf \mathcal{F} on M , $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{F})|_{f^{-1}(0)}$ is called the (j -th) *vanishing cycle sheaf* of \mathcal{F} . Here $\mathbf{R}^+ = \{t \in \mathbf{R}; t \geq 0\}$. This measures how the cohomology groups of \mathcal{F} change across the fibers of f . Its algebro-geometric version is studied by Grothendieck-Deligne ([D]).

5.2. Let (X, \mathcal{O}_X) be a complex manifold. Let $f: X \rightarrow \mathbf{R}$ be a C^∞ -map and consider the vanishing cycle sheaf $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$. Let s be the section of $f^{-1}(0) \rightarrow T^*X$ given by df . Then we have

PROPOSITION 5.2.1 ([KS1] § 3, [K2] § 4.2). $\mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$ has a structure of an $s^{-1}\mathcal{E}_X$ -module.

Let P be a differential operator. If $\sigma(P)$ does not vanish on $s(f^{-1}(0))$, then P has an inverse in $s^{-1}\mathcal{E}_X$ by Proposition 2.2.3. Therefore we obtain

COROLLARY 5.2.2. If $\sigma(P)|_{s(f^{-1}(0))} \neq 0$, then

$$P: \mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)} \rightarrow \mathcal{H}_{f^{-1}(\mathbf{R}^+)}^j(\mathcal{O}_X)|_{f^{-1}(0)}$$

is bijective.

5.3. More generally, let \mathcal{M} be a coherent \mathcal{D}_X -module, and set

$$\mathcal{F}^\bullet = \mathbf{R}\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X).$$

Then the preceding corollary shows that

$$\mathbf{R}\Gamma_{f^{-1}(\mathbf{R}^+)}(\mathcal{F}^\bullet)|_{f^{-1}(0)} = 0 \quad \text{if} \quad s(f^{-1}(0)) \cap \text{Ch}(\mathcal{M}) = \emptyset.$$

Here $\text{Ch}\mathcal{M}$ denotes the characteristic variety of \mathcal{M} .

5.4. To consider vanishing cycle sheaves is very near to the "microlocal" consideration. In this direction, see [K-S2].

§ 6. MICRO-DIFFERENTIAL OPERATORS AND THE SYMPLECTIC STRUCTURE ON THE COTANGENT BUNDLE

6.1. The ring \mathcal{E}_X is a non-commutative ring. This fact gives rise to new phenomena which are not shared by commutative rings such as the ring of