

# §7. Quantized Contact Transformations

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **32 (1986)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **11.08.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

An analytic subset  $V$  of  $T^*X$  is called *involutive* if  $f|_V = g|_V = 0$  implies  $\{f, g\}|_V = 0$ .

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

**THEOREM 6.3.2 ([G]).** *Let  $\mathcal{M}$  be a coherent  $\mathcal{E}_X$ -module defined on an open subset  $\Omega$  of  $T^*X$  and let  $\mathcal{L}$  be a  $\mathcal{E}_X(0)|_\Omega$ -module which is a union of coherent  $\mathcal{E}_X(0)$ -modules. Then  $V = \{p \in \Omega; \mathcal{L} \text{ is not coherent over } \mathcal{E}_X(0) \text{ on any neighborhood of } p\}$  is an involutive analytic subset of  $\Omega$ .*

**COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]).** *For any coherent  $\mathcal{E}_X$ -module  $\mathcal{M}$ ,  $\text{Supp } \mathcal{M}$  is involutive.*

Since any involutive subset has codimension less than or equal to  $\dim X$ , we have

**COROLLARY 6.3.4.** *The support of a coherent  $\mathcal{E}_X$ -module has codimension  $\leq \dim X$ .*

After some algebraic calculation, this implies

**THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5).** *For any point  $p \in T^*X$ ,  $\mathcal{E}_{X,p}$  has a global cohomological dimension  $\dim X$ .*

6.4. An analytic subset  $\Lambda$  of  $T^*X$  is called *Lagrangian* if  $\Lambda$  is involutive and  $\dim \Lambda = \dim X$ . A coherent  $\mathcal{E}_X$ -module is called *holonomic* if its support is Lagrangian.

## § 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of  $T^*X$  is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

*Definition 7.2.1.* Let  $X$  and  $Y$  be complex manifolds of the same dimension. A morphism  $\varphi$  from an open subset  $U$  of  $T^*X$  to  $T^*Y$  is called a *homogeneous symplectic transformation* if  $\varphi^*\theta_Y = \theta_X$ .

We can easily see the following

(7.2.1) If  $\varphi$  is a homogeneous symplectic transformation, then  $\varphi$  is a local isomorphism and is compatible with the action of  $\mathbf{C}^*$ .

(7.2.2) Assume  $Y = \mathbf{C}^n$  and let  $(y_1, \dots, y_n; \eta_1, \dots, \eta_n)$  be the coordinates of  $T^*Y$ , so that  $\theta_Y = \sum \eta_j dy_j$ .

Set  $p_j = \eta_j \circ \varphi$  and  $q_j = y_j \circ \varphi$ . Then we have

(7.2.3.1)  $\{p_j, p_k\} = \{q_j, q_k\} = 0, \{p_j, q_k\} = \delta_{j,k}$  for  $j, k = 1, \dots, n$ .

(7.2.3.2)  $p_j$  is homogeneous of degree 1 and  $q_j$  is homogeneous of degree 0 with respect to the fiber coordinates.

(7.2.4) Conversely assume that functions  $\{q_1, \dots, q_n, p_1, \dots, p_n\}$  on  $U \subset T^*X$  satisfy (7.2.3.1) and (7.2.3.2). Then the map  $\varphi: U \rightarrow T^*Y$ , given by

$$U \ni x \mapsto (q_1(x), \dots, q_n(x); p_1(x), \dots, p_n(x)) \in T^*Y,$$

is a homogeneous symplectic transformation. We call  $(q_1, \dots, q_n; p_1, \dots, p_n)$  a *homogeneous symplectic coordinate system*.

**THEOREM 7.2.2** ([SKK] Chap. II § 3.2, [K2] § 2.4, [Bj] Chap. 4 § 6).

Let  $\varphi: T^*X \supset U \rightarrow T^*Y$  be a homogeneous symplectic transformation, let  $p_X$  be a point of  $U$  and set  $p_Y = \varphi(p_X)$ . Then we have

- (a) There exists an open neighborhood  $U'$  of  $p_X$  and a  $\mathbf{C}$ -algebra isomorphism  $\Phi: \varphi^{-1}\mathcal{E}_Y|_{U'} \xrightarrow{\sim} \mathcal{E}_X|_{U'}$  (we call  $(\varphi, \Phi)$  a *quantized contact transformation*).
- (b) If  $\Phi: \varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$  is a  $\mathbf{C}$ -algebra homomorphism then for any  $m, \Phi$  gives an isomorphism  $\varphi^{-1}\mathcal{E}_Y(m) \xrightarrow{\sim} \mathcal{E}_X(m)|_U$ . Moreover the following diagram commutes:

$$\begin{array}{ccc} \varphi^{-1}\mathcal{E}_Y(m) & \xrightarrow{\Phi} & \mathcal{E}_X(m)|_U \\ \downarrow \sigma_m & & \downarrow \sigma_m \\ \varphi^{-1}\mathcal{O}_{T^*Y}(m) & \xrightarrow{\varphi^*} & \mathcal{O}_{T^*Y}(m)|_U \end{array}$$

- (c) Let  $\Phi$  and  $\Phi'$  be two  $\mathbf{C}$ -algebra homomorphisms  $\varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$ .

Then there exist  $\lambda \in \mathbf{C}$ , a neighborhood  $U'$  of  $p_X$  and  $P \in \Gamma(U; \mathcal{E}_X(\lambda))$  such that  $\sigma_\lambda(P)$  is invertible and

$$\Phi'(Q) = P\Phi(Q)P^{-1} \quad \text{for } Q \in \varphi^{-1}\mathcal{E}_Y|_{U'}.$$

Moreover  $\lambda$  is unique and  $P$  is unique up to constant multiple.

(d) Let  $Y = \mathbf{C}^n$  and let  $U$  be an open subset of  $T^*X$ .

If  $P_j \in \Gamma(U; \mathcal{E}_X(1))$  and  $Q_j \in \Gamma(U; \mathcal{E}_X(0))$  ( $1 \leq j \leq n$ ) satisfy

$$(7.2.5) \quad \begin{aligned} [P_j, P_k] &= [Q_j, Q_k] = 0 \\ [P_j, Q_k] &= \delta_{jk} \end{aligned}$$

then there exists a unique quantized contact transformation  $(\varphi, \Phi)$  such that

$$\varphi(p) = (\sigma_0(Q_1)(p), \dots, \sigma_0(Q_n)(p), \sigma_1(P_1)(p), \dots, \sigma_1(P_n)(p)),$$

and  $\Phi(y_j) = Q_j, \Phi(\partial_{y_j}) = P_j$ .

We call  $\{Q_1, \dots, Q_n, P_1, \dots, P_n\}$  quantized canonical coordinates.

7.3. We shall give several examples of quantized contact transformations.

*Example 7.3.1.* If  $P(\partial)$  is a constant coefficient micro-differential operator of order 1, then

$$(x_1 + [P, x_1], x_2 + [P, x_2], \dots, x_n + [P, x_n], \partial_{x_1}, \dots, \partial_{x_n})$$

gives quantized canonical coordinates.

*Example 7.3.2.* More generally if  $P$  is a micro-differential operator of order 1 and  $\exp tH_{\sigma_1(P)}$  exists, then  $\exp tP$  gives a quantized contact transformation  $\Phi_t$ , by solving the equation  $\frac{d}{dt}\Phi_t(Q) = [P, \Phi_t(Q)]$  with the initial condition  $\Phi_t(Q) = Q$  for  $t = 0$ .

*Example 7.3.3.* (Paraboloidal transformation [K2] p. 36). Set  $X = \mathbf{C}^{1+n} = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n\}$ ,

$$\Omega = \{(t, x; \tau, \xi) \in T^*X; \tau \neq 0\}, G = \text{Sp}(n; \mathbf{C})$$

$$= \{g \in \text{GL}(2n; \mathbf{C}); {}^t g J g = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For  $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ , let  $\Psi_g$  be the quantized contact transformation given by

$$\begin{aligned} \partial_x &\mapsto \alpha\partial_x - \beta x\partial_t \\ x &\mapsto \gamma\partial_x\partial_t^{-1} + \delta x \\ \partial_t &\mapsto \partial_t \\ t &\mapsto t + \frac{1}{2} \{ \langle \partial_x, {}^t\gamma\alpha\partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t\gamma\beta x \rangle \partial_t^{-1} \\ &\quad + \langle {}^t\gamma\beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t\delta\beta x \rangle \}. \end{aligned}$$

Then we have  $\Psi_{g_1}\Psi_{g_2} = \Psi_{g_1g_2}$ .

§ 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES  
(See [SKK])

8.1. External Tensor Product.

Let  $X$  and  $Y$  be complex manifolds and let  $p_1$  and  $p_2$  be the projections  $T^*(X \times Y) \rightarrow T^*X$  and  $T^*(X \times Y) \rightarrow T^*Y$ , respectively. Then  $\mathcal{E}_{X \times Y}$  contains  $p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y$  as a subring. For an  $\mathcal{E}_X$ -module  $\mathcal{M}$  and an  $\mathcal{E}_Y$ -module  $\mathcal{N}$ , we define the  $\mathcal{E}_{X \times Y}$ -module  $\mathcal{M} \hat{\otimes} \mathcal{N}$  by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \otimes_{p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y} (p_1^{-1}\mathcal{M} \otimes_{\mathbb{C}} p_2^{-1}\mathcal{N}).$$

Then one can easily see

PROPOSITION 8.1.1.

- (i)  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is an exact functor in  $\mathcal{M}$  and in  $\mathcal{N}$  and  $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp} \mathcal{M} \times \text{Supp} \mathcal{N}$ .
- (ii) If  $\mathcal{M}$  is  $\mathcal{E}_X$ -coherent and  $\mathcal{N}$  is  $\mathcal{E}_Y$ -coherent, then  $\mathcal{M} \hat{\otimes} \mathcal{N}$  is  $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold  $Y$  of a complex manifold  $X$  of codimension  $l$ , the sheaf  $\lim_{\rightarrow m} \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{I}^m, \mathcal{O}_X)$  has a natural structure of  $\mathcal{D}_X$ -module,

which is denoted by  $\mathcal{B}_{Y|X}$ . Here  $\mathcal{I}$  is the defining ideal of  $Y$ . The homomorphism  $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$  gives the canonical section  $c(Y, X)$  of  $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ . If we take local coordinates  $(x_1, \dots, x_n)$  of  $X$  such that  $Y$  is defined by  $x_1 = \dots = x_l = 0$ , then we have