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An analytic subset V of T^*X is called *involutive* if $f|_V = g|_V = 0$ implies $\{f, g\}|_V = 0$.

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

THEOREM 6.3.2 ([G]). *Let \mathcal{M} be a coherent \mathcal{E}_X -module defined on an open subset Ω of T^*X and let \mathcal{L} be a $\mathcal{E}_X(0)|_\Omega$ -module which is a union of coherent $\mathcal{E}_X(0)$ -modules. Then $V = \{p \in \Omega; \mathcal{L} \text{ is not coherent over } \mathcal{E}_X(0) \text{ on any neighborhood of } p\}$ is an involutive analytic subset of Ω .*

COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]). *For any coherent \mathcal{E}_X -module \mathcal{M} , $\text{Supp } \mathcal{M}$ is involutive.*

Since any involutive subset has codimension less than or equal to $\dim X$, we have

COROLLARY 6.3.4. *The support of a coherent \mathcal{E}_X -module has codimension $\leq \dim X$.*

After some algebraic calculation, this implies

THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5). *For any point $p \in T^*X$, $\mathcal{E}_{X,p}$ has a global cohomological dimension $\dim X$.*

6.4. An analytic subset Λ of T^*X is called *Lagrangian* if Λ is involutive and $\dim \Lambda = \dim X$. A coherent \mathcal{E}_X -module is called *holonomic* if its support is Lagrangian.

§ 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of T^*X is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

Definition 7.2.1. Let X and Y be complex manifolds of the same dimension. A morphism φ from an open subset U of T^*X to T^*Y is called a *homogeneous symplectic transformation* if $\varphi^*\theta_Y = \theta_X$.

We can easily see the following

(7.2.1) If φ is a homogeneous symplectic transformation, then φ is a local isomorphism and is compatible with the action of \mathbf{C}^* .

(7.2.2) Assume $Y = \mathbf{C}^n$ and let $(y_1, \dots, y_n; \eta_1, \dots, \eta_n)$ be the coordinates of T^*Y , so that $\theta_Y = \sum \eta_j dy_j$.

Set $p_j = \eta_j \circ \varphi$ and $q_j = y_j \circ \varphi$. Then we have

(7.2.3.1) $\{p_j, p_k\} = \{q_j, q_k\} = 0, \{p_j, q_k\} = \delta_{j,k}$ for $j, k = 1, \dots, n$.

(7.2.3.2) p_j is homogeneous of degree 1 and q_j is homogeneous of degree 0 with respect to the fiber coordinates.

(7.2.4) Conversely assume that functions $\{q_1, \dots, q_n, p_1, \dots, p_n\}$ on $U \subset T^*X$ satisfy (7.2.3.1) and (7.2.3.2). Then the map $\varphi: U \rightarrow T^*Y$, given by

$$U \ni x \mapsto (q_1(x), \dots, q_n(x); p_1(x), \dots, p_n(x)) \in T^*Y,$$

is a homogeneous symplectic transformation. We call $(q_1, \dots, q_n; p_1, \dots, p_n)$ a *homogeneous symplectic coordinate system*.

THEOREM 7.2.2 ([SKK] Chap. II § 3.2, [K2] § 2.4, [Bj] Chap. 4 § 6).

Let $\varphi: T^*X \supset U \rightarrow T^*Y$ be a homogeneous symplectic transformation, let p_X be a point of U and set $p_Y = \varphi(p_X)$. Then we have

- (a) There exists an open neighborhood U' of p_X and a \mathbf{C} -algebra isomorphism $\Phi: \varphi^{-1}\mathcal{E}_Y|_{U'} \xrightarrow{\sim} \mathcal{E}_X|_{U'}$ (we call (φ, Φ) a *quantized contact transformation*).
- (b) If $\Phi: \varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$ is a \mathbf{C} -algebra homomorphism then for any m, Φ gives an isomorphism $\varphi^{-1}\mathcal{E}_Y(m) \xrightarrow{\sim} \mathcal{E}_X(m)|_U$. Moreover the following diagram commutes:

$$\begin{array}{ccc} \varphi^{-1}\mathcal{E}_Y(m) & \xrightarrow{\Phi} & \mathcal{E}_X(m)|_U \\ \downarrow \sigma_m & & \downarrow \sigma_m \\ \varphi^{-1}\mathcal{O}_{T^*Y}(m) & \xrightarrow{\varphi^*} & \mathcal{O}_{T^*Y}(m)|_U \end{array}$$

- (c) Let Φ and Φ' be two \mathbf{C} -algebra homomorphisms $\varphi^{-1}\mathcal{E}_Y \rightarrow \mathcal{E}_X|_U$.

Then there exist $\lambda \in \mathbf{C}$, a neighborhood U' of p_X and $P \in \Gamma(U; \mathcal{E}_X(\lambda))$ such that $\sigma_\lambda(P)$ is invertible and

$$\Phi'(Q) = P\Phi(Q)P^{-1} \quad \text{for } Q \in \varphi^{-1}\mathcal{E}_Y|_{U'}.$$

Moreover λ is unique and P is unique up to constant multiple.

(d) Let $Y = \mathbf{C}^n$ and let U be an open subset of T^*X .

If $P_j \in \Gamma(U; \mathcal{E}_X(1))$ and $Q_j \in \Gamma(U; \mathcal{E}_X(0))$ ($1 \leq j \leq n$) satisfy

$$(7.2.5) \quad \begin{aligned} [P_j, P_k] &= [Q_j, Q_k] = 0 \\ [P_j, Q_k] &= \delta_{jk} \end{aligned}$$

then there exists a unique quantized contact transformation (φ, Φ) such that

$$\varphi(p) = (\sigma_0(Q_1)(p), \dots, \sigma_0(Q_n)(p), \sigma_1(P_1)(p), \dots, \sigma_1(P_n)(p)),$$

and $\Phi(y_j) = Q_j, \Phi(\partial_{y_j}) = P_j$.

We call $\{Q_1, \dots, Q_n, P_1, \dots, P_n\}$ quantized canonical coordinates.

7.3. We shall give several examples of quantized contact transformations.

Example 7.3.1. If $P(\partial)$ is a constant coefficient micro-differential operator of order 1, then

$$(x_1 + [P, x_1], x_2 + [P, x_2], \dots, x_n + [P, x_n], \partial_{x_1}, \dots, \partial_{x_n})$$

gives quantized canonical coordinates.

Example 7.3.2. More generally if P is a micro-differential operator of order 1 and $\exp tH_{\sigma_1(P)}$ exists, then $\exp tP$ gives a quantized contact transformation Φ_t , by solving the equation $\frac{d}{dt}\Phi_t(Q) = [P, \Phi_t(Q)]$ with the initial condition $\Phi_t(Q) = Q$ for $t = 0$.

Example 7.3.3. (Paraboloidal transformation [K2] p. 36). Set $X = \mathbf{C}^{1+n} = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n\}$,

$$\Omega = \{(t, x; \tau, \xi) \in T^*X; \tau \neq 0\}, G = \text{Sp}(n; \mathbf{C})$$

$$= \{g \in \text{GL}(2n; \mathbf{C}); {}^t g J g = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, let Ψ_g be the quantized contact transformation given by

$$\begin{aligned} \partial_x &\mapsto \alpha\partial_x - \beta x\partial_t \\ x &\mapsto \gamma\partial_x\partial_t^{-1} + \delta x \\ \partial_t &\mapsto \partial_t \\ t &\mapsto t + \frac{1}{2} \{ \langle \partial_x, {}^t\gamma\alpha\partial_x \rangle \partial_t^{-2} + \langle \partial_x, {}^t\gamma\beta x \rangle \partial_t^{-1} \\ &\quad + \langle {}^t\gamma\beta x, \partial_x \rangle \partial_t^{-1} + \langle x, {}^t\delta\beta x \rangle \}. \end{aligned}$$

Then we have $\Psi_{g_1}\Psi_{g_2} = \Psi_{g_1g_2}$.

§ 8. FUNCTORIAL PROPERTIES OF MICRO-DIFFERENTIAL MODULES
(See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \rightarrow T^*X$ and $T^*(X \times Y) \rightarrow T^*Y$, respectively. Then $\mathcal{E}_{X \times Y}$ contains $p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y$ as a subring. For an \mathcal{E}_X -module \mathcal{M} and an \mathcal{E}_Y -module \mathcal{N} , we define the $\mathcal{E}_{X \times Y}$ -module $\mathcal{M} \hat{\otimes} \mathcal{N}$ by

$$(8.1.1) \quad \mathcal{M} \hat{\otimes} \mathcal{N} = \mathcal{E}_{X \times Y} \otimes_{p_1^{-1}\mathcal{E}_X \otimes_{\mathbb{C}} p_2^{-1}\mathcal{E}_Y} (p_1^{-1}\mathcal{M} \otimes_{\mathbb{C}} p_2^{-1}\mathcal{N}).$$

Then one can easily see

PROPOSITION 8.1.1.

- (i) $\mathcal{M} \hat{\otimes} \mathcal{N}$ is an exact functor in \mathcal{M} and in \mathcal{N} and $\text{Supp}(\mathcal{M} \hat{\otimes} \mathcal{N}) = \text{Supp} \mathcal{M} \times \text{Supp} \mathcal{N}$.
- (ii) If \mathcal{M} is \mathcal{E}_X -coherent and \mathcal{N} is \mathcal{E}_Y -coherent, then $\mathcal{M} \hat{\otimes} \mathcal{N}$ is $\mathcal{E}_{X \times Y}$ -coherent.

8.2. For a complex submanifold Y of a complex manifold X of codimension l , the sheaf $\lim_{\rightarrow m} \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_X/\mathcal{I}^m, \mathcal{O}_X)$ has a natural structure of \mathcal{D}_X -module,

which is denoted by $\mathcal{B}_{Y|X}$. Here \mathcal{I} is the defining ideal of Y . The homomorphism $\mathcal{O}_Y \rightarrow \mathcal{E}xt_{\mathcal{O}_X}^l(\mathcal{O}_Y, \Omega_X^l) \rightarrow \Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega_X^l \otimes_{\mathcal{O}_X} \mathcal{B}_{Y|X}$. If we take local coordinates (x_1, \dots, x_n) of X such that Y is defined by $x_1 = \dots = x_l = 0$, then we have