§7. Quantized Contact Transformations

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An analytic subset V of T^*X is called *involutive* if $f|_V = g|_V = 0$ implies $\{f, g\}|_V = 0$.

The following theorem exhibits a phenomenon which has no analogue in the commutative case.

THEOREM 6.3.2 ([G]). *Let* $\mathcal{M}$ *be a coherent* $\mathcal{E}_X$*-module defined on an open subset* Ω *of* $\mathring{T}^*X$ *and let* $\mathcal{L}$ *be a* $\mathcal{E}_X(0)|_\Omega$*-module which is a union of coherent* $\mathcal{E}_X(0)$*-modules. Then* $V = \{p \in \Omega;\ \mathcal{L}$ is not coherent over $\mathcal{E}_X(0)$ on any neighborhood of $p\}$ *is an involutive analytic subset of* Ω.

COROLLARY 6.3.3 ([SKK] Chap. II, Theorem 5.3.2, [M]). *For any coherent* $\mathcal{E}_X$*-module* $\mathcal{M}$, Supp $\mathcal{M}$ *is involutive.*

Since any involutive subset has codimension less than or equal to dim X, we have

COROLLARY 6.3.4. *The support of a coherent* $\mathcal{E}_X$*-module has codimension* $\leqslant$ dim X.

After some algebraic calculation, this implies

THEOREM 6.3.5 ([SKK] Chap. II, Theorem 5.3.5). *For any point* $p \in T^*X$, $\mathcal{E}_{X,p}$ *has a global cohomological dimension* dim X.

6.4. An analytic subset Λ of T^*X is called *Lagrangean* if Λ is involutive and dim Λ = dim X. A coherent $\mathcal{E}_X$-module is called *holonomic* if its support is Lagrangean.

§ 7. QUANTIZED CONTACT TRANSFORMATIONS

7.1. In the previous section, we saw that the symplectic structure of T^*X is closely related to micro-differential operators via the relation of commutator and Poisson bracket. In this section, we shall explain another relation.

Definition 7.2.1. Let X and Y be complex manifolds of the same dimension. A morphism φ from an open subset U of T^*X to T^*Y is called a *homogeneous symplectic transformation* if $\varphi^*\theta_Y = \theta_X$.

We can easily see the following

(7.2.1) If φ is a homogeneous symplectic transformation, then φ is a local isomorphism and is compatible with the action of $\mathbf{C}^*$.

(7.2.2) Assume $Y = \mathbf{C}^n$ and let $(y_1, ..., y_n; \eta_1, ..., \eta_n)$ be the coordinates of T^*Y, so that $\theta_Y = \sum \eta_j dy_j$.

Set $p_j = \eta_j \circ \varphi$ and $q_j = y_j \circ \varphi$. Then we have

(7.2.3.1) $\{p_j, p_k\} = \{q_j, q_k\} = 0, \{p_j, q_k\} = \delta_{j,k}$ for $j, k = 1, ..., n$.

(7.2.3.2) p_j is homogeneous of degree 1 and q_j is homogeneous of degree 0 with respect to the fiber coordinates.

(7.2.4) Conversely assume that functions $\{q_1, ..., q_n, p_1, ..., p_n\}$ on $U \subset T^*X$ satisfy (7.2.3.1) and (7.2.3.2). Then the map $\varphi: U \to T^*Y$, given by

$$U \ni x \mapsto (q_1(x), ..., q_n(x); p_1(x), ..., p_n(x)) \in T^*Y ,$$

is a homogeneous symplectic transformation. We call $(q_1, ..., q_n; p_1, ..., p_n)$ a *homogeneous symplectic coordinate system.*

THEOREM 7.2.2 ([SKK] Chap. II § 3.2, [K2] § 2.4, [Bj] Chap. 4 § 6). *Let $\varphi: T^*X \supset U \to T^*Y$ be a homogeneous symplectic transformation, let p_X be a point of U and set $p_Y = \varphi(p_X)$. Then we have*

(a) *There exists an open neighborhood U' of p_X and a* **C***-algebra isomorphism $\Phi: \varphi^{-1}\mathscr{E}_Y|_{U'} \xrightarrow{\sim} \mathscr{E}_X|_{U'}$ (we call (φ, Φ) a quantized contact transformation).*

(b) *If $\Phi: \varphi^{-1}\mathscr{E}_Y \to \mathscr{E}_X|_U$ is a* **C***-algebra homomorphism then for any m, Φ gives an isomorphism $\varphi^{-1}\mathscr{E}_Y(m) \xrightarrow{\sim} \mathscr{E}_X(m)|_U$. Moreover the following diagram commutes:*

$$\begin{array}{ccc} \varphi^{-1}\mathscr{E}_Y(m) & \xrightarrow{\Phi} & \mathscr{E}_X(m)|_U \\ \downarrow \sigma_m & & \downarrow \sigma_m \\ \varphi^{-1}\mathscr{O}_{T^*Y}(m) & \xrightarrow{\varphi^*} & \mathscr{O}_{T^*Y}(m)|_U \end{array}$$

(c) *Let Φ and Φ' be two* **C***-algebra homomorphisms $\varphi^{-1}\mathscr{E}_Y \to \mathscr{E}_X|_U$.*

Then there exist $\lambda \in \mathbf{C}$, *a neighborhood* U' *of* p_X *and* $P \in \Gamma(U; \mathscr{E}_X(\lambda))$ *such that* $\sigma_\lambda(P)$ *is invertible and*

$$\Phi'(Q) = P\Phi(Q)P^{-1} \quad \textit{for} \quad Q \in \varphi^{-1}\mathscr{E}_Y|_{U'}.$$

Moreover λ *is unique and* P *is unique up to constant multiple.*

(d) *Let* $Y = \mathbf{C}^n$ *and let* U *be an open subset of* T^*X.

If $P_j \in \Gamma(U; \mathscr{E}_X(1))$ *and* $Q_j \in \Gamma(U; \mathscr{E}_X(0))$ $(1 \leqslant j \leqslant n)$ *satisfy*

$$[P_j, P_k] = [Q_j, Q_k] = 0 \tag{7.2.5}$$
$$[P_j, Q_k] = \delta_{jk}$$

then there exists a unique quantized contact transformation (φ, Φ) *such that*

$$\varphi(p) = (\sigma_0(Q_1)(p), \ldots, \sigma_0(Q_n)(p), \sigma_1(P_1)(p), \ldots, \sigma_1(P_n)(p)),$$

and $\Phi(y_j) = Q_j, \Phi(\partial_{y_j}) = P_j$.

We call $\{Q_1, \ldots, Q_n, P_1, \ldots, P_n\}$ quantized canonical coordinates.

7.3. We shall give several examples of quantized contact transformations.

Example 7.3.1. If $P(\partial)$ is a constant coefficient micro-differential operator of order 1, then

$$(x_1 + [P, x_1], x_2 + [P, x_2], \ldots, x_n + [P, x_n], \partial_{x_1}, \ldots, \partial_{x_n})$$

gives quantized canonical coordinates.

Example 7.3.2. More generally if P is a micro-differential operator of order 1 and $\exp tH_{\sigma_1(P)}$ exists, then $\exp tP$ gives a quantized contact transformation Φ_t, by solving the equation $\frac{d}{dt}\Phi_t(Q) = [P, \Phi_t(Q)]$ with the initial condition $\Phi_t(Q) = Q$ for $t = 0$.

Example 7.3.3. (Paraboloidal transformation [K2] p. 36). Set $X = \mathbf{C}^{1+n} = \{(t, x) \in \mathbf{C} \times \mathbf{C}^n\}$,

$$\Omega = \{(t, x; \tau, \xi) \in T^*X; \tau \neq 0\}, G = \mathrm{Sp}(n; \mathbf{C})$$
$$= \{g \in \mathrm{GL}(2n; \mathbf{C}); {}^t gJg = J\} \quad \text{with} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$, let Ψ_g be the quantized contact transformation given by

$$\partial_x \mapsto \alpha\partial_x - \beta x\partial_t$$

$$x \mapsto \gamma\partial_x\partial_t^{-1} + \delta x$$

$$\partial_t \mapsto \partial_t$$

$$t \mapsto t + \frac{1}{2}\{<\partial_x, {}^t\gamma\alpha\partial_x> \partial_t^{-2} + <\partial_x, {}^t\gamma\beta x> \partial_t^{-1}$$

$$+ <{}^t\gamma\beta x, \partial_x> \partial_t^{-1} + <x, {}^t\delta\beta x>\}.$$

Then we have $\Psi_{g_1}\Psi_{g_2} = \Psi_{g_1g_2}$.

§ 8. Functorial Properties of Micro-Differential Modules (See [SKK])

8.1. External Tensor Product.

Let X and Y be complex manifolds and let p_1 and p_2 be the projections $T^*(X \times Y) \to T^*X$ and $T^*(X \times Y) \to T^*Y$, respectively. Then $\mathscr{E}_{X \times Y}$ contains $p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y$ as a subring. For an $\mathscr{E}_X$-module $\mathscr{M}$ and an $\mathscr{E}_Y$-module $\mathscr{N}$, we define the $\mathscr{E}_{X \times Y}$-module $\mathscr{M} \hat{\otimes} \mathscr{N}$ by

$$\mathscr{M} \hat{\otimes} \mathscr{N} = \mathscr{E}_{X \times Y} \underset{p_1^{-1}\mathscr{E}_X \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{E}_Y}{\otimes} (p_1^{-1}\mathscr{M} \underset{\mathbf{C}}{\otimes} p_2^{-1}\mathscr{N}). \tag{8.1.1}$$

Then one can easily see

Proposition 8.1.1.

(i) *$\mathscr{M} \hat{\otimes} \mathscr{N}$ is an exact functor in $\mathscr{M}$ and in $\mathscr{N}$ and* $\operatorname{Supp}(\mathscr{M} \hat{\otimes} \mathscr{N}) = \operatorname{Supp} \mathscr{M} \times \operatorname{Supp} \mathscr{N}$.

(ii) *If $\mathscr{M}$ is $\mathscr{E}_X$-coherent and $\mathscr{N}$ is $\mathscr{E}_Y$-coherent, then $\mathscr{M} \hat{\otimes} \mathscr{N}$ is $\mathscr{E}_{X \times Y}$-coherent.*

8.2. For a complex submanifold Y of a complex manifold X of codimension l, the sheaf $\varinjlim_m \mathscr{E}xt^l_{\mathscr{O}_X}(\mathscr{O}_X/\mathscr{J}^m, \mathscr{O}_X)$ has a natural structure of $\mathscr{D}_X$-module, which is denoted by $\mathscr{B}_{Y|X}$. Here $\mathscr{J}$ is the defining ideal of Y. The homomorphism $\mathscr{O}_Y \to \mathscr{E}xt^l_{\mathscr{O}_X}(\mathscr{O}_Y, \Omega^l_X) \to \Omega^l_X \underset{\mathscr{O}_X}{\otimes} \mathscr{B}_{Y|X}$ gives the canonical section $c(Y, X)$ of $\Omega^l_X \underset{\mathscr{O}_X}{\otimes} \mathscr{B}_{Y|X}$. If we take local coordinates $(x_1, ..., x_n)$ of X such that Y is defined by $x_1 = ... = x_l = 0$, then we have