§9. Regularity Conditions (See [KK], [K-O])

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THEOREM 8.10.1. *Let* U_X *and* U_Y *be open subsets of* T^*X *and* T^*Y, *respectively. Let* $\mathcal{N}$ *be a coherent* $(\mathscr{E}_Y|_{U_Y})$*-module. Assume*

(i) $\rho_g\colon \tilde{\omega}_g^{-1}(\operatorname{Supp}\mathcal{N}) \cap \rho_g^{-1}(U_X) \to U_X$ *is a finite morphism.*

Then we have

(a) $\mathscr{T}or_j^{\tilde{\omega}_g^{-1}\mathscr{E}_Y}(\mathscr{E}_{X\leftarrow Y}, \tilde{\omega}_g^{-1}\mathcal{N}) = 0$ for $j \neq 0$.

(b) $\mathcal{M} = \rho_{g*}(\mathscr{E}_{X\leftarrow Y} \underset{\tilde{\omega}_g^{-1}\mathscr{E}_Y}{\otimes} \tilde{\omega}_g^{-1}\mathcal{N})|_{U_X}$ *is a coherent* $\mathscr{E}_X|_{U_X}$*-module.*

(c) $\operatorname{Supp}\mathcal{M} = \rho_g(\tilde{\omega}_g^{-1}\operatorname{Supp}\mathcal{N} \cap U_X)$.

§ 9. REGULARITY CONDITIONS (See [KK], [K-O])

9.1. Let us recall the notion of regular singularity of ordinary differential equations. Let $P(x, \partial) = \sum_{j \leq m} a_j(x)\partial^j$ be a linear differential operator in one variable x. We assume that the $a_j(x)$ are holomorphic on a neighborhood of $x = 0$. Then we say that the origin 0 is a regular singularity of $Pu = 0$ if

(*) $$\operatorname{ord}_{x=0} a_j(x) \geqslant \operatorname{ord}_{x=0} a_m(x) - (m-j)\,.$$

Here $\operatorname{ord}_{x=0}$ means the order of the zero. In this case, the local structure of the equation is very simple. In fact, the $\mathscr{D}_X$-module $\mathscr{D}_X/\mathscr{D}_X P$ is a direct sum of copies of the following modules:

$$\mathcal{O}_X = \mathscr{D}_X/\mathscr{D}_X\partial,\ \mathscr{B}_{\{0\}|X} = \mathscr{D}_X/\mathscr{D}_X x,\ \mathscr{D}_X/\mathscr{D}_X(x\partial-\lambda)^{m+1} \quad (\lambda\in\mathbf{C},\, m\in\mathbf{N}),$$
$$\mathscr{D}_X/\mathscr{D}_X(x\partial)^{m+1}x \quad (m\in\mathbf{N}),\ \mathscr{D}_X/\mathscr{D}_X\partial(x\partial)^{m+1} \quad (m\in\mathbf{N})\,.$$

If we denote by u the canonical generator, then we have $Pu = 0$. By multiplying either a power of ∂ or a power of x, we obtain

$$\sum_{j=0}^{N} b_j(x)\,(x\partial)^j u = 0$$

with $b_N(x) = 1$. Hence $\mathscr{F} = \sum_{j=0}^{\infty} \mathcal{O}(x\partial)^j u = \sum_{j=0}^{N-1} \mathcal{O}(x\partial)^j u$ is a coherent $\mathcal{O}$-sub-module of $\mathcal{M}$ which satisfies $(x\partial)\mathscr{F} \subset \mathscr{F}$. We shall generalize this property to the case of several variables.

9.2. Let X be a complex manifold, Ω an open subset of $\mathring{T}^*X$ and V a closed involutive complex submanifold of Ω. Let us define

$$\mathscr{J}_V = \{u \in \mathscr{E}(1)|_\Omega;\ \sigma_1(P)|_V = 0\}$$

and let $\mathscr{E}_V$ be the subring of $\mathscr{E}_X|_\Omega$ generated by $\mathscr{J}_V$. For a coherent $\mathscr{E}_X$-module $\mathscr{M}$, a coherent sub-$\mathscr{E}_X(0)$-module $\mathscr{L}$ of $\mathscr{M}$ is called a *lattice* of $\mathscr{M}$ if $\mathscr{M} = \mathscr{E}_X\mathscr{L}$. The following proposition is easily derived from the fact that $\mathscr{E}(0)$ is a Noetherian ring.

PROPOSITION 9.2.1 ([K-O] Theorem 1.4.7). *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X|_\Omega$-module. Then the following conditions are equivalent.*

(1) *For any point $p \in \Omega$, there is a lattice $\mathscr{M}_0$ of $\mathscr{M}$ on a neighborhood of p such that $\mathscr{J}_V\mathscr{M}_0 = \mathscr{M}_0$.*

(2) *For any open subset U of Ω and for any coherent $\mathscr{E}(0)$-submodule $\mathscr{L}$ of $\mathscr{M}|_U$, $\mathscr{E}_V\mathscr{L}$ is coherent over $\mathscr{E}(0)|_U$.*

Definition 9.2.2. If the equivalent conditions of the preceding proposition are satisfied, then we say that $\mathscr{M}$ has *regular singularities* along V.

Remark that if $\mathscr{M}$ has regular singularity along V, then the support of $\mathscr{M}$ is contained in V. Let us denote by $IR_V(\mathscr{M})$ the set of points p such that $\mathscr{M}$ has no regular singularities along V on any neighborhood of p.

The following theorem is an immediate consequence of Gabber's Theorem 6.3.2.

THEOREM 9.2.3. *$IR_V(\mathscr{M})$ is an involutive analytic subset of $\mathscr{M}$.*

In fact, if we take a lattice $\mathscr{M}_0$ of $\mathscr{M}$, then $T^*X \backslash IR_V(\mathscr{M})$ is the largest open subset on which $\mathscr{E}_V\mathscr{M}_0$ is coherent over $\mathscr{E}(0)$.

9.3. If an $\mathscr{E}$-module $\mathscr{M}$ has regular singularities along an involutive submanifold V then $\mathscr{M}$ is, roughly speaking, constant along the bicharacteristics of V. More precisely, let Y and Z be complex manifolds and $X = Y \times Z$. Let $z_0 \in Z$ and let j be the inclusion map $Y \hookrightarrow X$ by $y \mapsto (y, z_0)$. Then we have

THEOREM 9.3.1. *Let $\mathscr{M}$ be a coherent $\mathscr{E}_X$-module. Assume that $\mathscr{M}$ has regular singularities along $T^*Y \times T^*_ZZ$. Then $\mathscr{M}$ is isomorphic to $j^*\mathscr{M} \hat{\otimes} \mathscr{O}_Z$.*

Note that any involutive submanifold V of T^*X with $\theta_X|_V \neq 0$ is transformed by a homogeneous symplectic transformation to the form $T^*Y \times T^*_ZZ$.

9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

THEOREM 9.4.1. *Let* $\mathcal{M}$ *be a holonomic* $\mathscr{E}_X$*-module. Then the following conditions are equivalent.*

(i) *There exists a Lagrangean subvariety* Λ *such that* $\mathcal{M}$ *has regular singularities along* Λ.

(ii) *For any involutive subvariety* Λ *which contains* $\operatorname{Supp}\mathcal{M}$, $\mathcal{M}$ *has regular singularities along* Λ.

(iii) *There exists an open dense subset* Ω *of* $\operatorname{Supp}\mathcal{M}$ *such that* $\mathcal{M}$ *has regular singularities along* $\operatorname{Supp}\mathcal{M}$ *on* Ω.

If these equivalent conditions are satisfied, we say that $\mathcal{M}$ is a *regular holonomic* $\mathscr{E}_X$-module.

The following properties are almost immediate.

THEOREM 9.4.2.

(i) *Let* $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ *be an exact sequence of three coherent* $\mathscr{E}_X$*-modules. If two of them are regular holonomic then so is the third.*

(ii) *If* $\mathcal{M}$ *is regular holonomic, its dual* $\mathcal{M}^*$ *is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

THEOREM 9.4.3 ([KK] Theorem 6.1.3). *If* $\mathcal{M}$ *and* $\mathcal{N}$ *are regular holonomic* $\mathscr{E}_X$*-modules, then* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathcal{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \hat{\mathscr{E}}_X \underset{\mathscr{E}_X}{\otimes} \mathcal{N})$ *and* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathcal{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathcal{M}, \mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathcal{N})$ *are isomorphisms.*

§ 10. STRUCTURE OF REGULAR HOLONOMIC $\mathscr{E}$-MODULES (See [SKK], [KK])

10.1. Let Λ be a Lagrangean submanifold of T^*X. We define $\mathscr{J}_\Lambda$ and $\mathscr{E}_\Lambda$ as in § 9.2.

Then $\mathscr{E}_\Lambda(-1) = \mathscr{E}_\Lambda \cdot \mathscr{E}(-1)$ is a two-sided ideal of $\mathscr{E}_\Lambda$ and $\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$ is a sheaf of rings which contains $\mathscr{O}_\Lambda(0) = \mathscr{E}(0)/\mathscr{J}_\Lambda(-1)$, the sheaf of homogeneous functions on Λ.