Zeitschrift: L'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: INTRODUCTION TO MICROLOCAL ANALYSIS
Kapitel: §10. Structure of Regular Holonomic E-Modules (See [SKK], [KK])
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DOI: https://doi.org/10.5169/seals-55089

9.4. Noting that any nowhere dense closed analytic subset of a Lagrangean variety is never involutive, Theorem 9.2.3 implies the following theorem.

THEOREM 9.4.1. *Let* $\mathscr{M}$ *be a holonomic* $\mathscr{E}_X$*-module. Then the following conditions are equivalent.*

(i) *There exists a Lagrangean subvariety* Λ *such that* $\mathscr{M}$ *has regular singularities along* Λ.

(ii) *For any involutive subvariety* Λ *which contains* $\operatorname{Supp}\mathscr{M}$, $\mathscr{M}$ *has regular singularities along* Λ.

(iii) *There exists an open dense subset* Ω *of* $\operatorname{Supp}\mathscr{M}$ *such that* $\mathscr{M}$ *has regular singularities along* $\operatorname{Supp}\mathscr{M}$ *on* Ω.

If these equivalent conditions are satisfied, we say that $\mathscr{M}$ is a *regular holonomic* $\mathscr{E}_X$-module.

The following properties are almost immediate.

THEOREM 9.4.2.

(i) *Let* $0 \to \mathscr{M}' \to \mathscr{M} \to \mathscr{M}'' \to 0$ *be an exact sequence of three coherent* $\mathscr{E}_X$*-modules. If two of them are regular holonomic then so is the third.*

(ii) *If* $\mathscr{M}$ *is regular holonomic, its dual* $\mathscr{M}^*$ *is also regular holonomic.*

We just mention another analytic property of regular holonomic modules, which generalizes the fact that a formal solution of an ordinary differential equation with regular singularity converges.

THEOREM 9.4.3 ([KK] Theorem 6.1.3). *If* $\mathscr{M}$ *and* $\mathscr{N}$ *are regular holonomic* $\mathscr{E}_X$*-modules, then* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \hat{\mathscr{E}}_X \underset{\mathscr{E}_X}{\otimes} \mathscr{N})$ *and* $\mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}) \to \mathscr{E}xt^j_{\mathscr{E}_X}(\mathscr{M}, \mathscr{E}^\infty_X \underset{\mathscr{E}_X}{\otimes} \mathscr{N})$ *are isomorphisms.*

§ 10. STRUCTURE OF REGULAR HOLONOMIC $\mathscr{E}$-MODULES (See [SKK], [KK])

10.1. Let Λ be a Lagrangean submanifold of T^*X. We define $\mathscr{J}_\Lambda$ and $\mathscr{E}_\Lambda$ as in § 9.2.

Then $\mathscr{E}_\Lambda(-1) = \mathscr{E}_\Lambda \cdot \mathscr{E}(-1)$ is a two-sided ideal of $\mathscr{E}_\Lambda$ and $\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$ is a sheaf of rings which contains $\mathscr{O}_\Lambda(0) = \mathscr{E}(0)/\mathscr{J}_\Lambda(-1)$, the sheaf of homogeneous functions on Λ.

Let us take an invertible $\mathcal{O}_\Lambda$-module $\mathcal{L}$ such that $\mathcal{L}^{\otimes 2} \cong \omega_\Lambda \underset{\mathcal{O}_X}{\otimes} \omega_X^{\otimes -1}$. Such an $\mathcal{L}$ exists at least locally. For $P = P_1(x, \partial) + P_0(x, \partial) + \dots \in \mathcal{J}$ we define, for $\varphi \in \mathcal{O}_\Lambda$ and an invertible section s of $\mathcal{L}$,

$$L(P)(\varphi s) = \left\{ H_{P_1}(\varphi) + \frac{1}{2}\varphi \frac{L_{H_{P_1}}(s^{\otimes 2} \otimes dx)}{s^{\otimes 2} \otimes dx} + \left(P_0 - \frac{1}{2}\sum_{j=1}^{n} \frac{\partial^2 P_1}{\partial x_i \partial \xi_i} \right) \varphi \right\} s .$$

Here $dx = dx_1 \wedge \dots \wedge dx_n \in \omega_X$ and $s^{\otimes 2} \otimes dx$ is regarded as a section of ω_Λ. The Lie derivative $L_{H_{P_1}}$ of H_{P_1} operates on ω_Λ as the first order differential operators so that $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)$ is a section of ω_Λ and $L_{H_{P_1}}(s^{\otimes 2} \otimes dx)/s^{\otimes 2} \otimes dx$ is a function on Λ.

We thus obtain $L: \mathcal{J}_\Lambda \to \mathcal{E}nd_{\mathbf{C}}(\mathcal{L})$. Then this does not depend on the choice of local coordinate system and moreover it extends to the ring homomorphism $L: \mathcal{E}_\Lambda \to \mathcal{E}nd_{\mathbf{C}}(\mathcal{L})$. Since the image is contained in the differential endomorphism of $\mathcal{L}$, we obtain the ring homomorphism $L: \mathcal{E}_\Lambda \to \mathcal{L} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{D}_\Lambda \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{L}^{\otimes -1}$.

PROPOSITION 10.1.1. *By* L, $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$ *coincides with the subsheaf of* $\mathcal{L} \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{D}_\Lambda \underset{\mathcal{O}_\Lambda}{\otimes} \mathcal{L}^{\otimes -1}$ *consisting of differential endomorphisms of* $\mathcal{L}$ *homogeneous of degree* 0.

If we take

$$\mathcal{J}_\Lambda \in \vartheta = \vartheta_1(x, \partial) + \vartheta_0(x, \partial) + \dots$$

such that $d\vartheta_1 \equiv -\theta_X \bmod I_\Lambda \Omega^1$ and

$$\frac{1}{2}\sum \frac{\partial^2 \vartheta_1}{\partial x_i \partial \xi_i} \equiv \vartheta_0(x, \xi) \bmod \mathcal{J}_\Lambda$$

then $L(\vartheta)$ gives the Euler operator of $\mathcal{L}$. Such a ϑ is unique modulo $\mathcal{J}_\Lambda^2(-1) = \mathcal{E}_\Lambda(-1) \cap \mathcal{E}_X(1)$.

10.2. Let $\mathcal{M}$ be a regular holonomic $\mathcal{E}_X$-module whose support is Λ. Let $\mathcal{M}_0$ be a coherent sub-$\mathcal{E}_\Lambda$-module of M which generates $\mathcal{M}$. Such an $\mathcal{M}_0$ is called a *saturated lattice* of $\mathcal{M}$. Then $\bar{\mathcal{M}} = \mathcal{M}_0/\mathcal{E}(-1)\mathcal{M}_0$ is an $\mathcal{E}_\Lambda/\mathcal{E}_\Lambda(-1)$-module, which is coherent over $\mathcal{O}_\Lambda(0)$.

Since a coherent sheaf with integrable connection is locally free, we have

LEMMA 10.2.1. $\bar{\mathcal{M}}$ *is a locally free* $\mathcal{O}_\Lambda(0)$*-module of finite rank.*

Since ϑ belongs to the center of $\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)$, ϑ can be considered as an endomorphism of $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\bar{\mathscr{M}}, \mathscr{L})$, which is a locally constant sheaf on Λ. Its eigenvalues are called the *order* of $\mathscr{M}$ with respect to $\mathscr{M}_0$.

10.3. Let us take a section $G \subset \mathbf{C}$ of $\mathbf{C} \to \mathbf{C}/\mathbf{Z}$. Then there exists a unique saturated lattice $\mathscr{M}_0$ such that the orders of $\mathscr{M}$ with respect to $\mathscr{M}_0$ are contained in G (See [K4]). Then

$$\mathscr{F} = \mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\bar{\mathscr{M}}, \mathscr{L})$$

and

$$M = \exp 2\pi i\vartheta \in \mathscr{A}ut(\mathscr{F})$$

does not depend on the choice of G.

THEOREM 10.3.1 ([KK] Chapter I, § 3). *Assume that there exists an invertible $\mathscr{O}_\Lambda$-module $\mathscr{L}$ such that $\mathscr{L}^{\otimes 2} = \omega_\Lambda \otimes \omega_X^{\otimes -1}$. Then the category of regular holonomic $\mathscr{E}_X$-modules with support in Λ is equivalent to the category of $(\mathscr{F}, M)$'s where $\mathscr{F}$ is a locally constant $\mathbf{C}_\Lambda$-module and $M \in \mathscr{A}ut_{\mathbf{C}}(\mathscr{F})$.*

10.4. If $u \in \mathscr{M}$, then the solution to $L(P)\varphi = 0$ for $P \in \mathscr{E}_\Lambda$ with $Pu = 0$ is called a principal symbol of u and denoted by $\sigma(u)$. The homogeneous degree of $\sigma(u)$ is called the order of u. In the terminology of § 10.2, the principal symbol is a section of $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\mathscr{E}_\Lambda u/\mathscr{E}_\Lambda(-1)u, \mathscr{L})$ and the order is the eigenvalue of ϑ in $\mathscr{H}om_{\mathscr{E}_\Lambda/\mathscr{E}_\Lambda(-1)}(\mathscr{E}_\Lambda u/\mathscr{E}_\Lambda(-1)u, \mathscr{L})$.

10.4. When the characteristic variety is not smooth, we don't know much about the structure of holonomic systems. In this direction, we have

THEOREM 10.4.1 ([K-K] Theorem 1.2.2). *Let Z be a closed analytic subset of an open subset Ω of $T^*X, n = \dim X$, and let $\mathscr{M}$ and $\mathscr{N}$ be holonomic $\mathscr{E}_X|_\Omega$-modules.*

(i) *If* $\dim Z \leqslant n-1$, *then*

$$\Gamma(\Omega; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N})) \to \Gamma(\Omega \backslash Z, \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}))$$

is injective.

(ii) *If* $\dim Z \leqslant n-2$, *then*

$$\Gamma(\Omega; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N})) \to \Gamma(\Omega \backslash Z; \mathscr{H}om_{\mathscr{E}_X}(\mathscr{M}, \mathscr{N}))$$

is an isomorphism.

In particular if Supp $\mathcal{M} \subset \Lambda_1 \cup \Lambda_2$ *and if* $\dim(\Lambda_1 \cap \Lambda_2) \leqslant n-2$, *then* $\mathcal{M}$ *is a direct sum of two holonomic* $\mathcal{E}_X$*-modules supported on* Λ_1 *and* Λ_2, *respectively.*

Here is another type of theorem.

THEOREM 10.4.3 ([SKKO]). *Let* $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{J}$ *be a holonomic* $\mathcal{E}$*-module defined on a neighborhood of* $p \in T^*X$. *Assume* Supp $\mathcal{M} = \Lambda_1 \cup \Lambda_2$ *and*

(i) Λ_1, Λ_2 *and* $\Lambda_1 \cap \Lambda_2$ *are non-singular and* $\dim \Lambda_1 = \dim \Lambda_2 = n$, $\dim(\Lambda_1 \cap \Lambda_2) = n-1$.

(ii) $T_{p'}\Lambda_1 \cap T_{p'}\Lambda_2 = T_{p'}(\Lambda_1 \cap \Lambda_2)$ *for any* p' *in a neighborhood of* p *in* $\Lambda_1 \cap \Lambda_2$.

(iii) *The symbol ideal of* $\mathcal{J}$ *coincides with the ideal of functions vanishing on* $\Lambda_1 \cup \Lambda_2$.

Setting $k = \mathrm{ord}_{\Lambda_1} u - \mathrm{ord}_{\Lambda_2} u - 1/2$, *we have*

(a) $\mathcal{M}$ *has a non-zero quotient supported on* $\Lambda_1 \Leftrightarrow \mathcal{M}$ *has a non-zero submodule supported on* $\Lambda_2 \Leftrightarrow k \in \mathbf{Z}$.

(b) $\mathcal{M}_p$ *is a simple* $\mathcal{E}_p$*-module* $\Leftrightarrow k \notin \mathbf{Z}$.

Sketch of the proof. By a quantized contact transformation, we can transform p, Λ_1, Λ_2 and $\mathcal{J}$ as follows:

$$p = (0, dx_1)$$

$$\Lambda_1 = \{(x, \xi); x_1 = \xi_2 = \dots = \xi_n = 0\}$$

$$\Lambda_2 = \{(x, \xi); x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}$$

$$\mathcal{J} = \mathcal{E}(x_1\partial_1 - \lambda) + \mathcal{E}(x_2\partial_2 - \mu) + \sum_{j>2} \mathcal{E}\partial_j$$

In this case, we can easily check the theorem.

§ 11. APPLICATION TO THE b-FUNCTION (see [SKKO])

11.1. As one of the most successful application of microlocal analysis, we shall sketch here how to calculate the b-function of a function under certain conditions.