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TREES, TAIL WAGGING AND GROUP PRESENTATIONS

by M. A. ARMSTRONG

The Bass-Serre theorem gives a presentation for a group of automorphisms of a tree. Like all good theorems it has attracted considerable attention and there are now several proofs available [4], [3], [1]. Our goal is a natural elementary proof which makes maximal use of the geometry of the tree.

1. GRAPHS

A *graph* X consists of two sets E (directed edges) and V (vertices) and two functions

$$E \to E, \quad e \mapsto \bar{e}$$

$$E \to V \times V, \quad e \mapsto (i(e), t(e))$$

which satisfy $\bar{\bar{e}} = e$, $\bar{e} \neq e$ and $i(\bar{e}) = t(e)$ for each $e \in E$. The vertices $i(e)$, $t(e)$ are the initial and terminal vertices of the directed edge e, and $\bar{e}$ is the reverse of e. Henceforth we refer to directed edges simply as edges.

A path in X joining vertex u to vertex v is an ordered string of edges $e_1 e_2 \dots e_n$ such that $i(e_1) = u$, $i(e_{k+1}) = t(e_k)$ for $1 \leqslant k \leqslant n-1$, and $t(e_n) = v$. If $v = u$ we have a circuit. A path of the form $e\bar{e}$ is a *round trip* and a circuit which does not contain any round trips will be called a loop. If any two distinct vertices may be joined by a path then the graph is connected. A *tree* is a connected graph which does not contain any loops.

Let X be a tree. A path in X is a *geodesic* if it does not contain any round trips. Given distinct vertices u, v of X there is a *unique* geodesic $\overrightarrow{uv}$ which joins u to v.

An action of a group G on a graph X is an action of G on E and on V such that $g\bar{e} = \overline{ge}$, $i(ge) = gi(e)$, $t(ge) = gt(e)$ and $ge \neq \bar{e}$ for each $e \in E$. Because group elements are not allowed to reverse edges we have a quotient graph X/G. When G acts on X we shall often say that G is a *group of automorphisms of* X.

We adopt the usual notation whereby G_x denotes the stabilizer of a vertex x. If $g \in G$ happens to fix x we write g_x for the element g thought of as a member of G_x. Of course G_e denotes the stabilizer of the edge e. If x is a vertex of e then G_e is a subgroup of G_x.

Suppose G acts on a tree X. If $g \in G$ fixes the vertices u, v then it must fix the whole geodesic $\overrightarrow{uv}$, since otherwise the image of $\overrightarrow{uv}$ under g would be a second geodesic from u to v.

2. Lifting edges

Let G be a group of automorphisms of a tree X. Choose a maximal tree M in X/G and lift it [4, Proposition I.14] to a subtree T of X. The vertices of T form a set of representatives for the action of G on the vertices of X. For each pair of edges f, $\bar{f}$ from $X/G - M$ select one, say f, and lift it to an edge e of X which has its initial vertex x in T. Exactly one vertex z of T lies in the same orbit as $t(e)$ and we choose an element γ_f from G that maps z onto $t(e)$. We can now lift $\bar{f}$ to $(\gamma_f)^{-1}\bar{e}$. This has its initial vertex z in T and $\gamma_{\bar{f}} = (\gamma_f)^{-1}$ sends the vertex x of T to its terminal vertex (Figure 1). Finally we extend the correspondence $f \to \gamma_f$ over the edges of M by setting $\gamma_f = 1$ (the identity element of G) whenever $f \in M$.

The *Bass-Serre theorem* [4, Theorem I.13] gives the following presentation for G.

(a) *Generators.* The elements of all the G_w where w is a vertex of T and the γ_f where f is an edge of X/G.

(b) *Relations.* The internal relations of each stabilizer G_w together with

$$\gamma_f = 1 \text{ if } f \text{ is an edge of } M,$$

$$\gamma_{\bar{f}} = (\gamma_f)^{-1} \text{ and}$$

$$\gamma_{\bar{f}}\, g_x \gamma_f = (\gamma_{\bar{f}}\, g\, \gamma_f)_z \text{ where } e \text{ is the chosen lift of } f \text{ and } g \in G_e.$$

(If f is an edge of M then $z = t(e)$ and the final relation reduces to $g_x = g_z$ whenever $g \in G_e$).

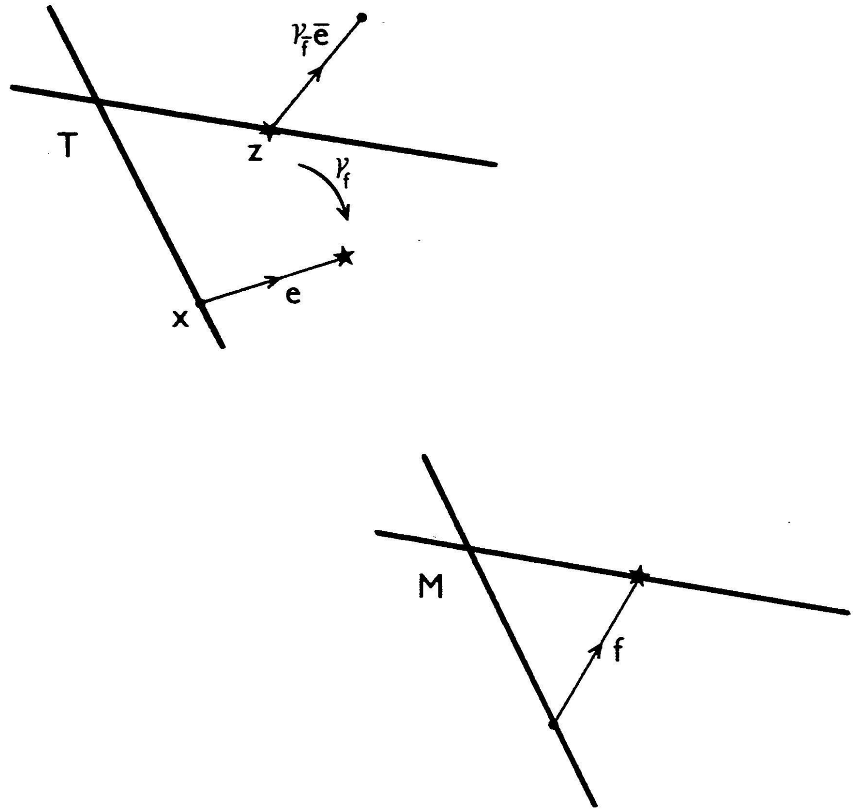

FIGURE 1

3. TAIL WAGGING

With the notation established above let $*G_w$ denote the free product of the stablizers of the vertices of T, and F the free group generated by symbols λ_f, one for each edge f of X/G. Let R be the normal consequence in $(*G_w)*F$ of the words

$$\lambda_f \quad (f \text{ an edge of } M),$$

$$\lambda_{\bar f}\,\lambda_f \quad \text{and}$$

$$\lambda_{\bar f}\, g_x \lambda_f (\gamma_{\bar f}\, g\, \gamma_f)_z^{-1}$$

We shall produce an isomorphism

$$\psi : G \to [(\ast G_w) \ast F]/R .$$

Choose a vertex v of T as base point. If $g \in G$ fixes v set

$$\psi(g) = g_v R$$

where as usual g_v is the element g interpreted as a member of G_v. If g moves v then it sends it outside T because no two vertices of T lie in the same orbit. Let $e_1 e_2 \ldots e_n$ be the geodesic which joins v to gv and suppose e_m is the first edge that is *not* in T. The path $e_m e_{m+1} \ldots e_n$ will be called the *tail* of $\overrightarrow{v\,gv}$. Let x_1 be the initial vertex of e_m. Project e_m into X/G to give an edge f_1. The canonical lift e^1 of f_1 into X has its initial vertex in T, so $i(e^1) = x_1$. Choose an element $a_{x_1} \in G_{x_1}$ which sends e^1 to e_m. Let

$$e_k^1 = (\gamma_{\bar{f}_1} a_{x_1}^{-1}) e_k$$

for $m+1 \leqslant k \leqslant n$, and replace $e_1 e_2 \ldots e_n$ by the new path $e_{m+1}^1 e_{m+2}^1 \ldots e_n^1$. We call this process *tail wagging*. Our new path begins at

$$z_1 = t(\gamma_{\bar{f}_1} e^1) = i(e_{m+1}^1)$$

which is a vertex of T and ends at $(\gamma_{\bar{f}_1} a_{x_1}^{-1} g)v$, see Figure 2. We walk along it to the first point x_2 where it quits T and repeat the above

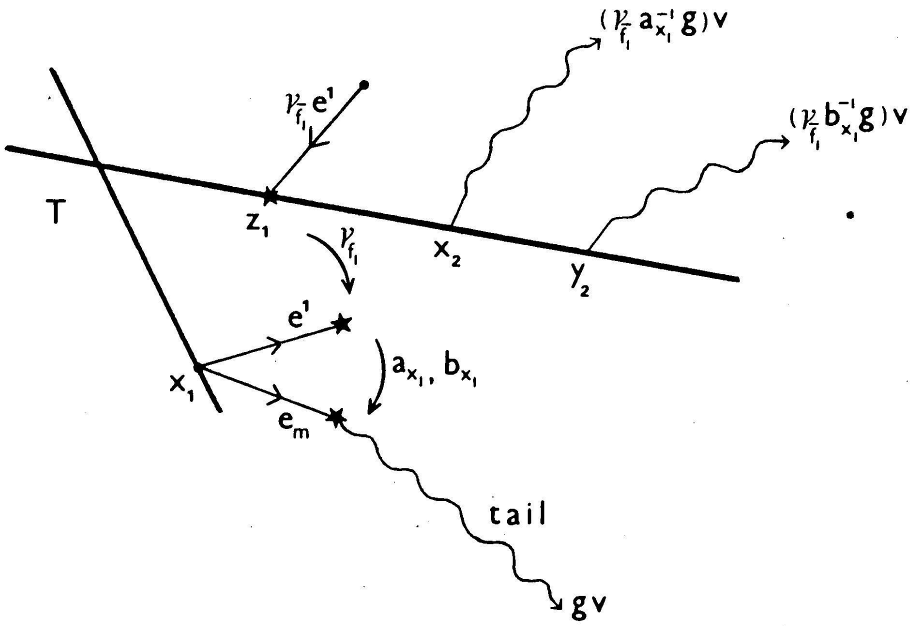

FIGURE 2

procedure. Since we shorten the tail at each step we eventually obtain a path which lies entirely in T and ends at say

$$(\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_2} a_{x_2}^{-1} \gamma_{\bar{f}_1} a_{x_1}^{-1} g)v .$$

Then $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g$ must fix v, say $\gamma_{\bar{f}_r} a_{x_r}^{-1} \dots \gamma_{\bar{f}_1} a_{x_1}^{-1} g = a_v \in G_v$. We now have

$$g = a_{x_1} \gamma_{f_1} \dots a_{x_r} \gamma_{f_r} a_v$$

and we somewhat optimistically define

$$\psi(g) = a_{x_1} \lambda_{f_1} \dots a_{x_r} \lambda_{f_r} a_v R .$$

4. An inefficient choice

Is ψ well defined? The geodesic from v to gv is certainly unique, as is the first point x_1 where it leaves T and its first edge e_m outside T. Both the edge e^1 and the group element γ_{f_1} are now determined by our original construction. The only ambiguity at this stage is the choice of the element $a_{x_1} \in G_{x_1}$ which maps e^1 to e_m. A different choice b_{x_1} will give a path from z_1 to $(\gamma_{\bar{f}_1} b_{x_1}^{-1} g)v$ which leaves T for the first time at say y_2. The first edge outside T will project to an edge f'_2 of X/G and so on until eventually we have g expressed as

$$g = b_{x_1} \gamma_{f_1} b_{y_2} \gamma_{f'_2} \dots b_{y_s} \gamma_{f'_s} b_v .$$

We must show that $a_{x_1} \lambda_{f_1} a_{x_2} \lambda_{f_2} \dots a_{x_r} \lambda_{f_r} a_v$ and $b_{x_1} \lambda_{f_1} b_{y_2} \lambda_{f'_2} \dots b_{y_s} \lambda_{f'_s} b_v$ determine the same left coset of R in $(\ast G_w) \ast F$.

Agree to select a_{x_1} from G_{x_1} so that the tail of the resulting path is as *long* as possible. Continue in this way selecting $a_{x_2}, a_{x_3} \dots$ so as to maximise the length of the tail at each stage. We shall compare any other set of choices with this rather inefficient selection.

Both a_{x_1} and b_{x_1} map e^1 to e_m, so $c = a_{x_1}^{-1} b_{x_1}$ must fix e^1. Also, due to our particular selection of a_{x_1}, the geodesic from z_1 to x_2 is left fixed by $\gamma_{\bar{f}_1} c \gamma_{f_1}$. Therefore

$$\begin{aligned} & b_{x_1}\,\lambda_{f_1}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v\,R \\ = {} & a_{x_1}\,\lambda_{f_1}\,\lambda_{\bar{f}_1}\,a_{x_1}^{-1}\,b_{x_1}\,\lambda_{f_1}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R \\ = {} & a_{x_1}\,\lambda_{f_1}\,\lambda_{\bar{f}_1}\,c_{x_1}\,\lambda_{f_1}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R \\ = {} & a_{x_1}\,\lambda_{f_1}(\gamma_{\bar{f}_1}c\gamma_{f_1})_{z_1}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R \\ = {} & a_{x_1}\,\lambda_{f_1}(\gamma_{\bar{f}_1}c\gamma_{f_1})_{x_2}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R \\ = {} & a_{x_1}\,\lambda_{f_1}\,a'_{x_2}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R \end{aligned}$$

where $a'_{x_2} = (\gamma_{\bar{f}_1}c\gamma_{f_1})_{x_2}$. If x_2 happens to equal y_2 then we simplify this further to

$$a_{x_1}\,\lambda_{f_1}\,a''_{x_2}\,\lambda_{f_2}\,b_{y_3}\,\lambda_{f'_3}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R$$

where a''_{x_2} is the product $a'_{x_2}\,b_{y_2}$ in G_{x_2}. We now compare a_{x_2} with a'_{x_2} if $x_2 \neq y_2$, noting that $\gamma_{f_2} = 1$ in this case, or with a''_{x_2} if $x_2 = y_2$, and repeat the process. Eventually we obtain

$$b_{x_1}\,\lambda_{f_1}\,b_{y_2}\,\lambda_{f'_2}\,...\,b_{y_s}\,\lambda_{f'_s}\,b_v R = a_{x_1}\,\lambda_{f_1}\,a_{x_2}\,\lambda_{f_2}\,...\,a_{x_r}\,\lambda_{f_r}\,a''_v\,R\,.$$

As $g = a_{x_1}\,\gamma_{f_1}\,...\,a_{x_r}\,\gamma_{f_r}\,a_v = a_{x_1}\,\gamma_{f_1}\,...\,a_{x_r}\,\gamma_{f_r}\,a''_v$ we see that $a''_v = a_v$. This completes the proof that ψ is well defined.

5. Nearest fixed points

To show ψ is a homomorphism we shall verify

$$\psi(hg) = \psi(h)\psi(g)$$

under the assumption that h either leaves some vertex of T fixed or is one of the elements γ_f. This is sufficient because the elements of the G_w (w a vertex of T) together with the γ_f (f an edge of $X/G-M$) form a set of generators for G.

Suppose h fixes the vertex w of T. Walk along the geodesic $\overrightarrow{vw}$ and let x be the first vertex we meet which is left fixed by h. Then $\overrightarrow{vx}$ is contained in T, and $\overrightarrow{vx}$ followed by $h(\overleftarrow{xv})$ is the geodesic from v to hv. This quits T for the first time at x and we see that

$$\psi(h) = h_x R\,.$$

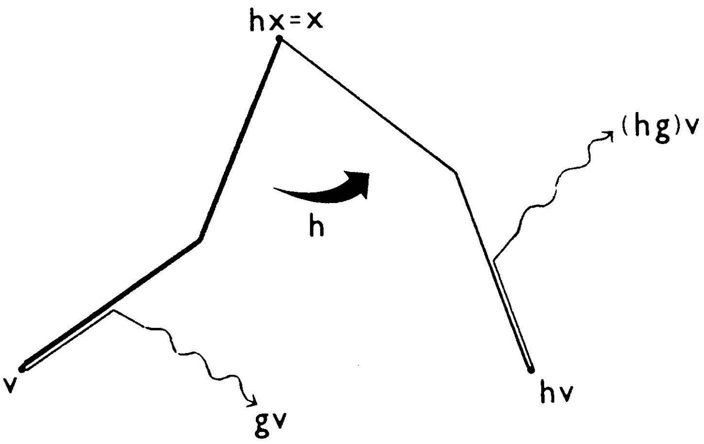

FIGURE 3

Using the geodesic from v to gv we have $\psi(g) = a_{x_1}\lambda_{f_1}\ldots a_{x_r}\lambda_{f_r}a_vR$ in the usual way. Therefore

$$\psi(h)\psi(g) = h_x a_{x_1}\lambda_{f_1}\ldots a_{x_r}\lambda_{f_r}a_vR\,.$$

In order to compute $\psi(hg)$ we need the geodesic from v to $(hg)v$. We can construct this as follows, take $\overrightarrow{v\,hv}$ followed by the image of $\overrightarrow{v\,gv}$ under h and remove any round trips.

If $\overrightarrow{v\,gv}$ does not contain all of $\overrightarrow{vx}$ (Figure 3) then $\overrightarrow{v(hg)v}$ leaves T for the first time at x. A tail wag of $\overrightarrow{v(hg)v}$ using h_x^{-1} leads us to a path which has the same tail as $\overrightarrow{v\,gv}$, then the process continues as for g. Thus

$$\psi(hg) = h_x a_{x_1}\lambda_{f_1}\ldots a_{x_r}\lambda_{f_r}a_vR = \psi(h)\psi(g)\,.$$

Otherwise $\overrightarrow{v\,gv}$ contains all of $\overrightarrow{vx}$ (Figure 4) and we split the argument into three cases.

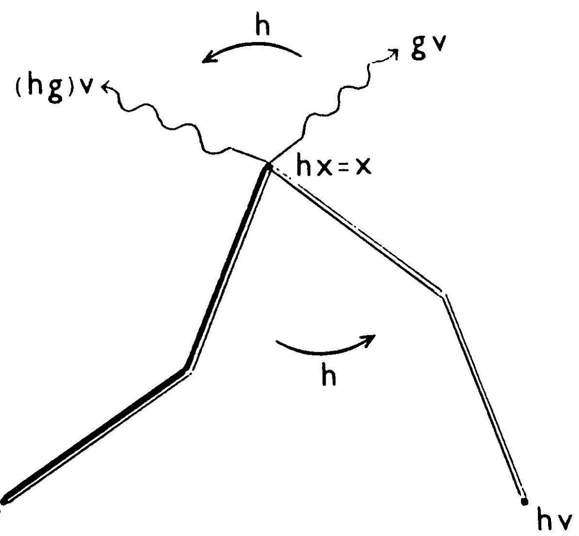

FIGURE 4

(a) $\overrightarrow{v\,gv}$ stays in T for at least one more edge after x. Then $\overrightarrow{v(hg)v}$ must leave T at x. As above, a first choice of h_x^{-1} leads to a path with the same tail as $\overrightarrow{v\,gv}$.

(b) $\overrightarrow{v\,gv}$ and $\overrightarrow{v(hg)v}$ both leave T at x. Then $x_1 = x$ and we write a_x instead of a_{x_1}. A first tail wag of $\overrightarrow{v(hg)v}$ using $\gamma_{\bar{f}_1}(h_x\, a_x)^{-1}$ produces the same path as a first tail wag of $\overrightarrow{v\,gv}$ using $\gamma_{\bar{f}_1}\, a_x^{-1}$. Thus

$$\psi(hg) = h_x\, a_x\, \lambda_{f_1}\, a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R = \psi(h)\psi(g)\,.$$

(c) $\overrightarrow{v\,gv}$ leaves T at x, but $\overrightarrow{v(hg)v}$ stays in T for at least one more edge after x. Then $x_1 = x$, $\gamma_{f_1} = 1$ and we may as well equate a_{x_1} with h_x^{-1}. A first tail wag of $\overrightarrow{v\,gv}$ using h_x gives a path with the same tail as $\overrightarrow{v(hg)v}$. Thus

$$\begin{aligned}\psi(hg) &= a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R\\ &= h_x\, h_x^{-1}\, a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R\\ &= \psi(h)\psi(g)\,.\end{aligned}$$

Suppose finally that $h = \gamma_f$ for some edge f of $X/G - M$. As usual e is the chosen lift of f into X with $x = i(e) \in T$ and $z = t(\gamma_{\bar{f}}\, e)$. Let $y = i(\gamma_{\bar{f}}\, e)$. The geodesic from v to $\gamma_f v$ is made up of $\overrightarrow{vx}$ followed by e followed by $\gamma_f(\overrightarrow{zv})$. This leaves T for the first time at x and a single tail wag using $\gamma_{\bar{f}}$ produces $\overrightarrow{zv}$. Therefore

$$\psi(\gamma_f) = \lambda_f R\,.$$

To obtain the geodesic from v to $(\gamma_f g)v$ we follow $\overrightarrow{v\gamma_f v}$ by $\gamma_f(\overrightarrow{v\,gv})$ and then remove any round trips (Figure 5). If $\overrightarrow{v\,gv}$ does not contain $\overrightarrow{vy}$, then $\overrightarrow{v(\gamma_f g)v}$ leaves T for the first time at x and a single tail wag using $\gamma_{\bar{f}}$

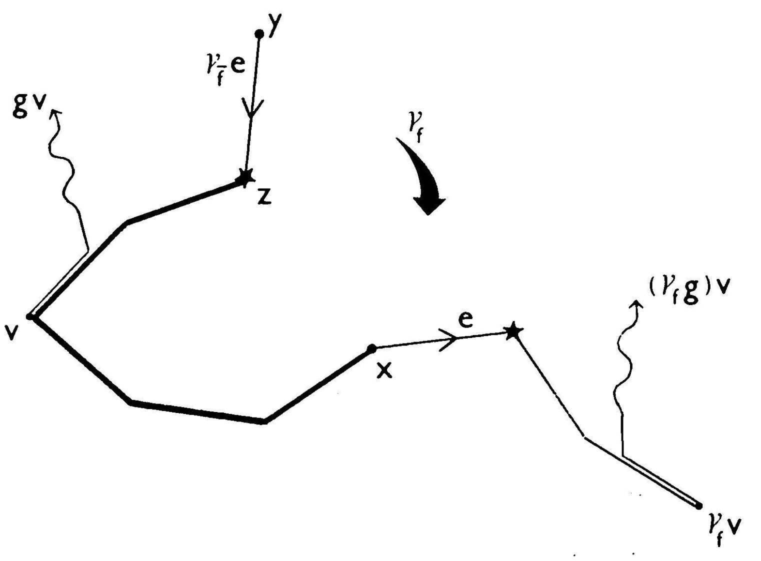

FIGURE 5

produces a path with the same tail as $\overrightarrow{v\, gv}$. The process then continues as for g and

$$\psi(\gamma_f g) = \lambda_f\, a_{x_1}\, \lambda_{f_1} \dots a_{x_r}\, \lambda_{f_r}\, a_v R = \psi(\gamma_f)\psi(g)\,.$$

Otherwise $\overrightarrow{v\, gv}$ contains $\overrightarrow{vy}$. Then $x_1 = z$, $\gamma_{f_1} \doteq \gamma_{\bar f}$ and we may as well take $a_{x_1} = 1$. A first tail wag of $\overrightarrow{v\, gv}$ using γ_f leaves a path with the same tail as $\overrightarrow{v(\gamma_f g)v}$. Thus

$$\begin{aligned}\psi(\gamma_f g) &= a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R \\ &= \lambda_f\, \lambda_{\bar f}\, a_{x_2}\, \lambda_{f_2} \dots a_{x_r}\, \lambda_{f_r}\, a_v R \\ &= \psi(\gamma_f)\psi(g)\,.\end{aligned}$$

This completes the proof that ψ is a homomorphism.

Our construction of ψ ensures that if $\psi(g) = R$ then $g = 1$. So ψ is injective. The cosets $h_w R$ (w a vertex of T and $h(w) = w$) and $\lambda_f R$ (f an edge of X/G) together generate $[(* G_w) * F]/R$. Now $\psi(h) = h_x R$ where x is the nearest fixed point of h to v. But h fixes all of $\overrightarrow{xw}$ so

$$\psi(h) = h_x R = h_w R\,.$$

Also

$$\psi(\gamma_f) = \lambda_f R\,.$$

Therefore the image of ψ is all of $[(* G_w) * F]/R$ and we have shown that ψ is an isomorphism.

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