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A paradox in the achievements of 3-dimensional topology between 1965 and 1985 is the following: Knot theory was gradually embodied in the more general theory of 3-dimensional manifolds. Classifications were attempted, and sometimes attained by using very refined geometrical tools such as the Waldhausen-Jaco-Shalen-Johanson theory on the embeddings of Seifert manifolds in a Haken manifold. And yet, these refined methods could not cope with simple questions related to knot projections. In fact, during this period, the old time point of view, using projections, was almost forgotten (except by a few people, for instance John Conway).

Today, Jones polynomials and more precisely L. Kauffman's very clever and very elementary way of looking at the one-variable polynomial $V_K(t)$ have put again knot projections under the spot-light. The one-variable polynomial is the main ingredient in the proofs of several of Tait's conjectures which have remained unproved for more than a century.

This paper is devoted to a presentation of these recent achievements, mainly due to V. Jones, L. Kauffman and K. Murasugi.

We shall give the definition and prove some of the properties of the two-variable Laurent polynomial $P(K) \in \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ associated with every oriented link K. The approach chosen here is that of V. Jones and A. Ocneanu. Another approach which uses the notion of skein invariance is due independently to many mathematicians: P. Freyd, D. Yetter, J. Hoste, W. Lickorish, K. Millett, J. Prztycki and P. Traczyk. Although we do discuss skein invariance in this paper, we do not go into the question of using it to define the polynomial $P(K)$.

As many mathematicians have worked simultaneously on various aspects of the definition of the polynomial, it is difficult to give proper credit to everyone. We apologize in advance for any missing ascription. We hope all will agree that V. Jones has been the one pioneer who got the subject started.

§ 2. Link diagrams

A link K in S^3 (or R^3) is a 1-dimensional compact smooth manifold without boundary. We shall use $r = r(K)$ for the number of components of K. A knot is a link with one component.

Most of the time K will be oriented.

Two oriented links K, K' are *ambient isotopic* if there exists a diffeomorphism $h: S^3 \to S^3$ of degree $+1$, such that $h(K) = K'$ and $h_{|K}$ is also of degree $+1$ on each component.

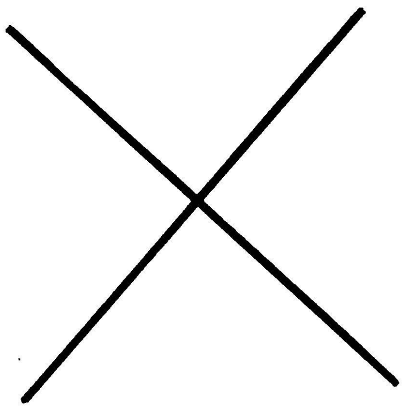
We will denote by $\mathscr{L}$ the set of ambient isotopy classes of oriented links.

A *link projection* is a generic immersion of a (finite) disjoint union of circles into the plane. (No triple point, transverse crossings only.)

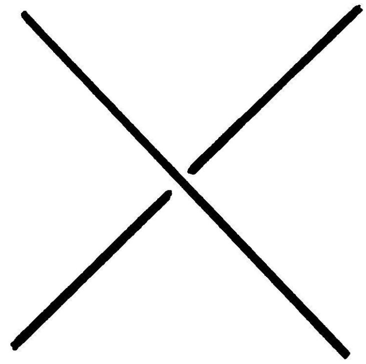
If $K \subset R^3$ is a link, an affine projection p of K on a plane $V \subset R^3$ gives a link projection if $p_{|K}$ is a generic immersion.

It is possible to recapture the isotopy class of K from the projection by specifying at each crossing point a choice of one of the two branches, singling out the branch which overcrosses the other.

A *link diagram* is a link projection together with such a choice of over/under crossing at each crossing point:



Part of a link projection



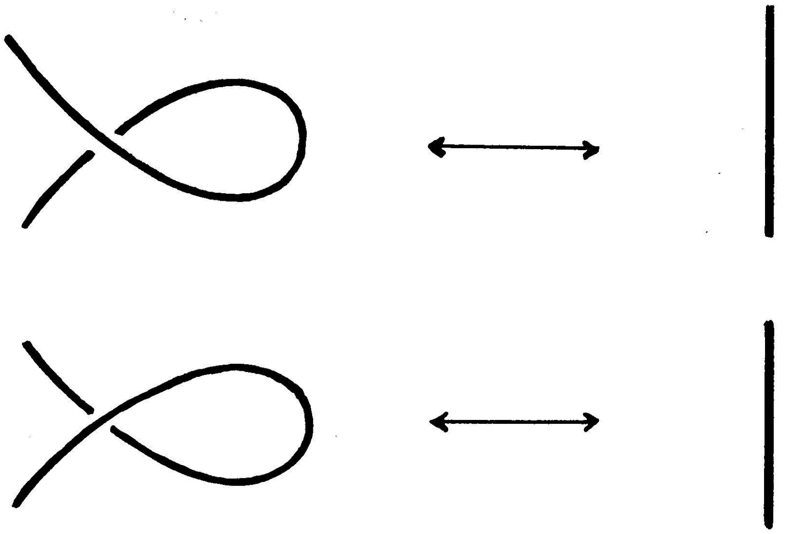
Part of a link diagram

A link diagram gives rise to a well defined ambient isotopy class of link in 3-space and every link isotopy class can be obtained in this way.

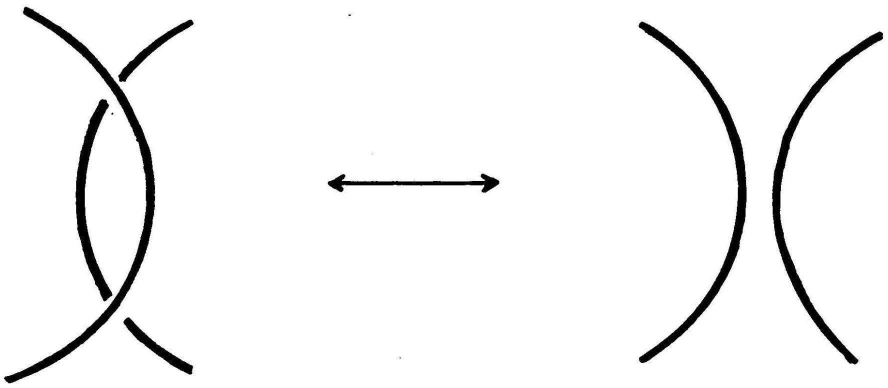
Of course many different link diagrams can give rise to isotopic links. This ambiguity is resolved by the notion of Reidemeister move:

The Reidemeister moves on link diagrams are the moves shown in the following pictures, for all possible orientations.

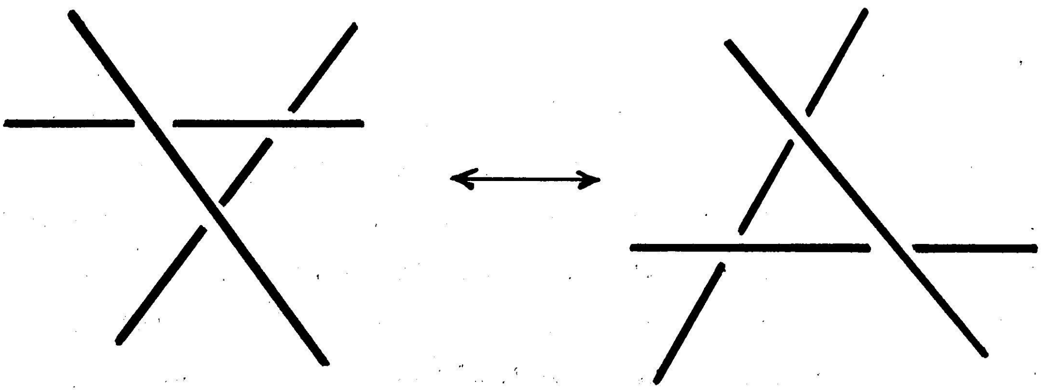
(i)



(ii)



(iii)



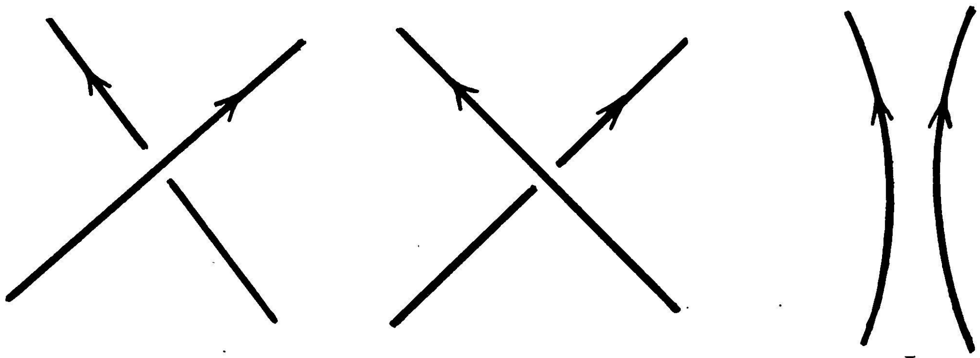
We shall take for granted without proof the classical

THEOREM. *Two oriented link diagrams represent ambient isotopic oriented links if and only if one can pass from one to the other by a finite sequence of Reidemeister moves.*

This theorem is the basis of combinatorial knot theory.

Another notion on link diagrams which will be of crucial importance in the sequel is that of *skein invariance*, due to J. Conway.

We say that 3 oriented links L_+, L_- and L_0 are *skein related* if they have diagrams which are identical except in the neighborhood of one crossing point where they look respectively as follows:



L_+ L_- L_0

Now, let $\mathscr{L}$ be the set of ambient isotopy classes of oriented links in R^3, and let A be a commutative ring. We say that a link invariant $P: \mathscr{L} \to A$ is a *linear skein invariant* if

(1) $P(\bigcirc) = 1$, where $\bigcirc$ denotes the 1-component unknot.

(2) There exist 3 invertible elements $a_+, a_-, a_0 \in A$ such that whenever L_+, L_-, L_0 are skein related, then $a_+P(L_+) + a_-P(L_-) + a_0P(L_0) = 0$.

Our objective is to define a skein invariant $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ with values in the ring of Laurent polynomials in 2 variables l, m. (Standing perhaps for Lickorish and Millett.) The elements a_+, a_-, a_0 will be respectively $a_+ = l, a_- = l^{-1}, a_0 = m$.

It will turn out that P is the universal linear skein invariant.