§3. Uniqueness and universality theorems

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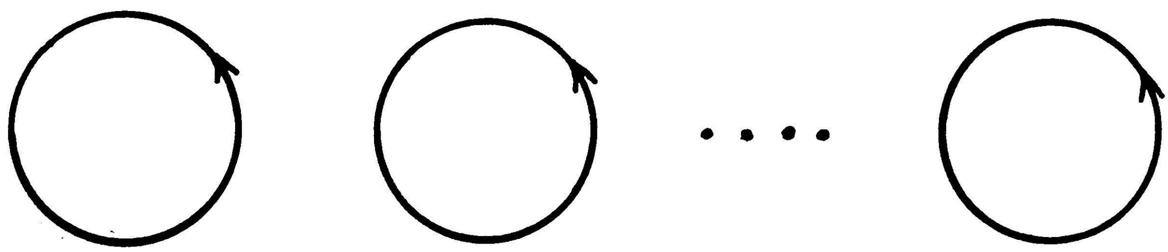
§ 3. UNIQUENESS AND UNIVERSALITY THEOREMS

We prove:

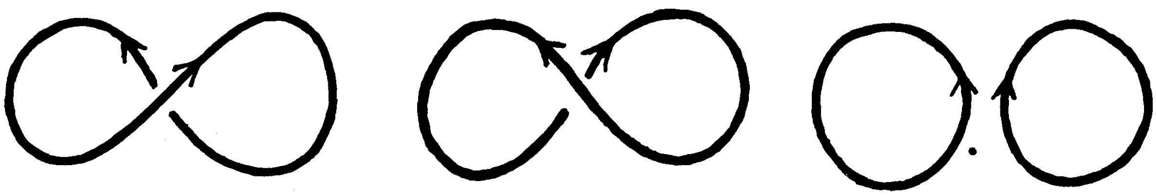
THEOREM 3.1. *If* $P: \mathscr{L} \to A$ *is a skein invariant, it is uniquely determined by the coefficients* a_+, a_- *and* a_0 *of the skein invariance relation.*

Proof. First note that $P(\bigcirc^r) = \left(- \frac{a_+ + a_-}{a_0} \right)^{r-1}$, where $\bigcirc^r$ denotes the unlink with r components.



(r circles)

Starting from $P(\bigcirc) = 1$, and the skein related link diagrams



L_+ L_- L_0

we see that

$$a_+ P(\bigcirc) + a_- P(\bigcirc) + a_0 P(\bigcirc^2) = 0,$$

and thus

$$P(\bigcirc^2) = - \frac{a_+ + a_-}{a_0}.$$

Adding $r - 1$ unknotted (and unlinked) disjoint components to each link in the above picture gives the desired formula by induction on r.

To prove the theorem, we shall use the following remark:

LEMMA 3.2. *For every link projection, there is a choice of over/under crossing at each crossing point which produces the unlink* $\bigcirc^r$.

Proof. The projection is a regular immersion of a disjoint union of circles. Order these circles arbitrarily. Now, running along the images of the circles, one after the other, declare that each new crossing is an underpass. (This involves the choice of a starting point on each circle such that its image is not a crossing point.)

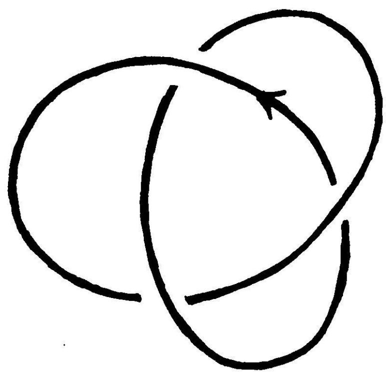
The link corresponding to this choice of crossings is $\bigcirc^r$. Indeed, it is clear that the various components are stacked, one above the other, in their chosen order, and are thus unlinked. Furthermore, it is easy to see that each component bounds a disk, and therefore is the unknot.

As a consequence of this lemma, if L is an arbitrary link diagram, there is a sequence of changes of over/under choices at the crossing points which carries the diagram into a diagram of the unlink $\bigcirc^r$ with the same number r of components.

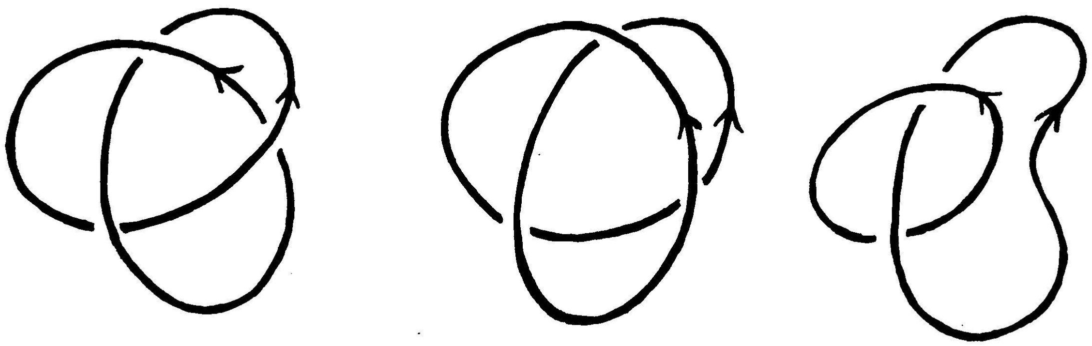
For each over/under change L_+, L_-, we get the first two terms of a skein relation (in a definite order). The third member L_0 of the skein related diagrams L_+, L_-, L_0 has one less crossing than L_+ and L_-. Thus, if we define the complexity of a link diagram to be the pair (N, S), where S is the number of crossing changes needed to get the unlink, and N is the number of crossings, and if we order the pairs by $(N, S) < (N', S')$ if $N < N'$ or $N = N'$ and $S < S'$ (alphabetical order), then we see by induction on the complexity that $P(L)$ is completely determined by $P(\bigcirc)$ and the skein invariance $a_+P(L_+) + a_-P(L_-) + a_0P(L_0) = 0$.

This proves theorem (3.1).

Example. Let T_+ be the right handed trefoil:



Use the skein relation

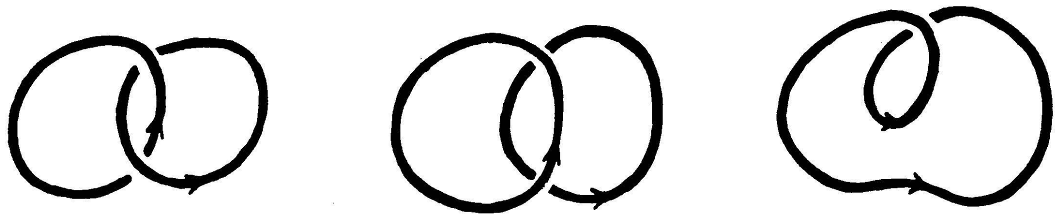


$T_+ = L_+$ $\quad$ $L_- = \bigcirc$ $\quad$ L_0

We get

$$a_+ P(T_+) + a_- + a_0 P(L_0) = 0 .$$

Then, another use of a skein relation



$L_0^+ = L_0$ $\quad$ $L_0^- = \bigcirc^2$ $\quad$ $L_0^0 = \bigcirc$

yields the formula

$$a_+ P(L_0) + a_- \cdot \left(- \frac{a_+ + a_-}{a_0} \right) + a_0 = 0 .$$

Solving for $P(T_+)$ in the two equations gives the result:

$$P(T_+) = -2a_- a_+^{-1} - a_-^2 a_+^{-2} + a_+^{-2} a_0^2 .$$

Of course, we do not know yet if P calculated in this way is well defined. But if it is well defined, $P(T_+)$ must be given by the above formula.

Now, let $A = \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ be the ring of Laurent polynomials in 2 variables l, m. Suppose $P : \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ is a skein invariant satisfying

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$$

for any 3 skein related link diagrams L_+, L_- and L_0.

THEOREM 3.3. *If such a* P *exists it is universal in the following sense:*

(1) P *determines a unique skein invariant*

$$T: \mathscr{L} \to \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$$

satisfying

$$xT(L_+) + yT(L_-) + zT(L_0) = 0$$

for every triple of skein related link diagrams L_+, L_- *and* L_0.

(2) *Moreover, if* $P_A: \mathscr{L} \to A$ *is any skein invariant with respect to three invertible elements* $a_+, a_-, a_0 \in A$ *as above, then* $s(T(K)) = P_A(K)$ *for all* $K \in \mathscr{L}$, *where* $s: \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}] \to A$ *is the obvious map determined by* $s(x) = a_+$, $s(y) = a_-$ *and* $s(z) = a_0$.

For the proof of this theorem, the crucial fact is the following assertion.

LEMMA 3.4. *Let* $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ *be a skein invariant as above, i.e.* $P(\bigcirc) = 1$, *and* $lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0$, *if* L_+, L_- *and* L_0 *are skein related. Then, each monomial* $l^a m^b$ *occuring (with non-zero coefficient) in* $P(K)$ *satisfies*

$$a \equiv b \bmod 2\,.$$

The proof of this lemma will actually show that

$$a \equiv b \equiv r(K) - 1 \bmod 2$$

for each monomial $l^a m^b$ of $P(K)$, where $r(K)$ is the number of connected components of K.

Proof of the lemma. True for the unknot, and more generally for the unlink with r components, since

$$P(\bigcirc^r) = \left(-\frac{l+l^{-1}}{m}\right)^{r-1}$$

as we have seen earlier.

Now, suppose that L_+, L_- and L_0 are 3 skein related link diagrams. Then, we have

$$P(L_+) = -l^{-2}P(L_-) - l^{-1}mP(L_0).$$

Hence, the claim follows by induction on the complexity of the link diagram, observing that $r(L_+) = r(L_-) = r(L_0) \pm 1$.

This completes the proof of lemma 3.4.

Now, given the skein invariant $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$, define $T: \mathscr{L} \to \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}]$ by replacing each monomial $l^a m^b$ of $P(K)$ by $x^i y^j z^k$, where

$$k = b,$$
$$i + j + k = 0,$$
$$i - j = a,$$

i.e. $i = 1/2(a-b), j = -1/2(a+b), k = b$.

By the above assertion ($a \equiv b \bmod 2$), T is a Laurent polynomial in x, y, z.

Perhaps more explicitly, we have

$$T(x, y, z) = P\big((x/y)^{1/2}, z \cdot (xy)^{-1/2}\big).$$

Observe that T is homogeneous of degree 0. This certainly is a necessary condition for T to be a skein invariant. (Exercise!)

It is clear that $T(\bigcirc) = 1$. We have to verify that

$$xT(L_+) + yT(L_-) + zT(L_0) = 0,$$

if L_+, L_- and L_0 are skein related.

Substituting $(x/y)^{1/2}$ for l and $z \cdot (xy)^{-1/2}$ for m in the relation

$$lP(L_+) + l^{-1}P(L_-) + mP(L_0) = 0,$$

we obtain

$$(x/y)^{1/2}T(L_+) + (y/x)^{1/2}T(L_-) + z(xy)^{-1/2}T(L_0) = 0$$

which yields the desired formula after multiplying by $(xy)^{1/2}$.

Further, if $P_A: \mathscr{L} \to A$ is any skein invariant (with respect to invertible elements a_+, a_-, a_0 of some commutative ring A) and if we define $s: \mathbf{Z}[x, x^{-1}, y, y^{-1}, z, z^{-1}] \to A$ by $s(x) = a_+$, $s(y) = a_-$, $s(z) = a_0$, then $s(T(L)) = P_A(L)$ follows for all link diagrams L by uniqueness, since both sT and P_A satisfy the same skein invariance with respect to a_+, a_-, a_0.

The existence of $P: \mathscr{L} \to \mathbf{Z}[l, l^{-1}, m, m^{-1}]$ will be proved in § 6, after some preliminaries on Hecke algebras in the next two paragraphs.