4. Symmetries of the Hopf fibrations with fibre S^3

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PROPOSITION 3.1. *The Hopf 3-spheres on* S^{4n-1} *are parallel to one another.*

The proof is similar to that of Proposition 1.1 for Hopf circles; it uses the fact that scalar multiplication by i, j and k are isometries of H^n. Alternatively, it will follow, as in Remark 1 above, from Proposition 4.2 a.
QED

The Riemannian metric on HP^{n-1} which makes the Hopf projection $S^{4n-1} \to HP^{n-1}$ into a Riemannian submersion is known as the *canonical metric* on HP^{n-1}. The canonical metric on HP^1 makes it into a round 4-sphere of radius 1/2. This follows by the same argument given in Proposition 1.2 for the case $H: S^1 \hookrightarrow S^3 \to CP^1$.

4. SYMMETRIES OF THE HOPF FIBRATIONS WITH FIBRE S^3

We now investigate the symmetries of the Hopf fibration

$$H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}.$$

The symplectic group

$$Sp(n) = Gl(n, H) \cap O(4n)$$

consists of quaternionically linear maps which are also rigid. Since these maps take quaternionic lines to quaternionic lines, they must be symmetries of the above Hopf fibration.

There are other symmetries. For each unit quaternion v, consider the action of right scalar multiplication by v on H^n,

$$R_v(u_1, ..., u_n) = (u_1 v, ..., u_n v).$$

This map is certainly *not* H-linear, since

$$R_v[(u_1, ..., u_n)w] = (u_1 w v, ..., u_n w v),$$

while

$$[R_v(u_1, ..., u_n)] w = (u_1 v w, ..., u_n v w).$$

Nevertheless, R_v takes each quaternionic line in H^n to itself. Thus the group S^3 of unit quaternions, acting on H^n from the right, must also be counted among the symmetries of our Hopf fibration.

Since the symplectic group $Sp(n)$ acts on S^{4n-1} from the left, while the group S^3 of unit quaternions acts from the right, these two actions commute. The actions also overlap, because they both contain multiplication by -1. Hence they combine to give an action of the group

$$\frac{Sp(n) \times S^3}{2}$$

on S^{4n-1}, where this group is obtained from $Sp(n) \times S^3$ by dividing out by the two-element subgroup consisting of the identity and the antipodal map. The following lemma asserts that there are no further symmetries.

PROPOSITION 4.1. *The group* G *of all symmetries of the Hopf fibration* H *is*

$$G = \frac{Sp(n) \times S^3}{2}.$$

Let g be a symmetry of the Hopf fibration, i.e., a rigid map of H^n taking quaternionic lines to quaternionic lines. Composing g with an appropriate element of $Sp(n)$, we can arrange that the new g be invariant on each quaternionic coordinate line $0 \times \ldots \times H \times \ldots \times 0$.

We claim this new g is orientation preserving on $H \times 0 \times \ldots \times 0$. Suppose not. Then composing it with appropriate elements of $Sp(n)$ and S^3, we can further arrange that $g(u, \ldots) = (\bar{u}, \ldots)$. Here we use the fact that left and right multiplication by unit quaternions generates the group $SO(4)$. Since g takes quaternionic lines to quaternionic lines, we must have

$$g(u, u, \ldots) = (\bar{u}, m\bar{u}, \ldots), \qquad \text{for some} \qquad m \neq 0.$$

Then for any s,

$$g(u, su, \ldots) = (\bar{u}, m\overline{(su)}, \ldots) = (\bar{u}, m\,\bar{u}\,\bar{s}, \ldots).$$

As u varies, these image points must also fill out a quaternionic line, hence $m\,\bar{u}\,\bar{s} = t\,\bar{u}$. Putting $u = 1$, we get $t = m\,\bar{s}$. Thus $m\,\bar{u}\,\bar{s} = m\,\bar{s}\,\bar{u}$. Cancelling the m, we get $\bar{u}\,\bar{s} = \bar{s}\,\bar{u}$. Since both u and s are arbitrary, this is impossible, establishing the claim.

Thus g is orientation preserving on $H \times 0 \times \ldots \times 0$, and we compose it with appropriate elements of $Sp(n)$ and S^3 so as to make it the identity there. Then we again use the fact that g takes quaternionic lines to quaternionic lines to conclude that

$$g(u, u, \ldots, u) = (u, m_2 u, \ldots, m_n u).$$

Hence

$$g(u_1, u_2, ..., u_n) = (u_1, m_2 u_2, ..., m_n u_n),$$

so the current version of g must lie in $Sp(n)$. QED

Remark. Note that all the symmetries are orientation preserving, since the group G is connected.

Let $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ denote our current Hopf fibration, and let us orient the fibres in a consistent fashion. The next proposition shows that this fibration is highly symmetric, yet slightly less so than the Hopf fibrations by circles.

PROPOSITION 4.2. *Let* $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ *be a Hopf fibration. Then*

a) *The only symmetries of* H *inducing the identity on the base space are the right multiplications by unit quaternions. This is just a 3-parameter subgroup of the 6-parameter group* $O(4)$ *of all rigid motions of a fibre.*

b) *If* P *and* Q *are any two fibres, then any preassigned orientation preserving rigid motion of* P *onto* Q *can be extended to a symmetry of* H. *But no orientation reversing one can.*

c) *The group of symmetries acts transitively on* S^{4n-1}, *and in particular acts transitively on fibres.*

It follows easily from the non-commutativity of the quaternions that the only transformations in $Sp(n)$ which take each quaternionic line to itself are $\pm Id$. Then a) follows immediately from the description of the symmetry group given in Proposition 4.1.

Even the subgroup $Sp(n)$ of G acts transitively on S^{4n-1}, and c) follows.

To prove b), we can now assume that P and Q both coincide with the unit 3-sphere on $H \times 0 \times ... \times 0$. Then left and right multiplication by unit quaternions takes this fibre to itself, and generates $SO(4)$. No orientation reversing transformation of this fibre can be achieved, since the group of symmetries is connected. This proves b). QED

Remarks. 1) Note that the existence of symmetries of H taking each fibre to itself and acting transitively on a given fibre shows that these fibres must be parallel.

2) Also note that a symmetry of $H: S^3 \hookrightarrow S^{4n-1} \to HP^{n-1}$ induces an

isometry of the base space HP^{n-1} in its canonical metric. It is easy to check that when $n = 2$, every orientation preserving isometry of the base $HP^1 = S^4(1/2)$ can be produced this way, while no orientation reversing one can (since the group is connected). We remark without proof that *all* isometries of HP^{n-1}, $n > 2$, can be produced this way, and that they are all orientation preserving.

5. Normed Division Algebras and the Cayley Numbers

In order to describe the Hopf fibration $H: S^7 \hookrightarrow S^{15} \to S^8$ in the next section, we first review here some facts about normed division algebras and the arithmetic of Cayley numbers. More can be found in two excellent references, [Cu] and [H-L, pp. 140-145].

A *normed division algebra B* is a finite dimensional algebra over the reals R, with multiplicative unit 1, and equipped with an inner product $< , >$ whose associated norm $| \ |$ satisfies

$$|xy| = |x|\,|y| \quad \text{for all} \quad x, y \in B .$$

By Hurwitz' Theorem ([Hu 1], 1898), a proof of which we will outline here, every normed division algebra is isomorphic to either the reals R, the complex numbers C, the quaternions H or the Cayley numbers Ca. Actually, what Hurwitz proved is that normed division algebras can only occur in dimensions 1, 2, 4 and 8. He stated the corresponding uniqueness result without proof. In [Hu 2], published in 1923 after his death, Hurwitz credits E. Robert [Ro] with writing out the details of the uniqueness argument in a 1912 Zurich thesis.

Now let B denote a given normed division algebra. Let $\operatorname{Re} B$ denote the one-dimensional linear subspace spanned by the identity 1, and $\operatorname{Im} B$ the orthogonal complement of $\operatorname{Re} B$. Then each $x \in B$ has a unique orthogonal decomposition,

$$x = x_1 + x', \quad x_1 \in \operatorname{Re} B \quad \text{and} \quad x' \in \operatorname{Im} B ,$$

into its real and imaginary parts. *Conjugation* in B is defined by:

$$\bar{x} = x_1 - x' .$$

Here are some basic facts about arithmetic in any normed division algebra B: