

Zeitschrift: L'Enseignement Mathématique
Herausgeber: Commission Internationale de l'Enseignement Mathématique
Band: 32 (1986)
Heft: 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

Artikel: THE GEOMETRY OF THE HOPF FIBRATIONS
Autor: Gluck, Herman / Ziller, Wolfgang

Bibliographie
DOI: <https://doi.org/10.5169/seals-55085>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. [Siehe Rechtliche Hinweise.](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. [Voir Informations légales.](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. [See Legal notice.](#)

Download PDF: 22.12.2024

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

It will be sufficient to look only at the symmetries of H which take the fibre $L_0 = \{(u, 0)\}$ to itself, and hence are of the form $(u, v) \mapsto (A(u), B(v))$. We already know that there must be a $C \in SO(8)$ such that $B(mu) = C(m) A(u)$ for all $m, u \in Ca$. To show that G is a nontrivial double covering of $SO(9)$, we must find a *loop* of C 's which lifts to a *non-loop* of (A, B) 's.

This can be done by using the Moufang identities, just as in the proof of the Triality Principle. Recall from that proof that if x is an imaginary Cayley number of unit length, then $A = L_x$, $B = -L_x$ and $C = L_x R_x$ "works", that is, $-L_x(mu) = L_x R_x(m) L_x(u)$. Now let x describe a semi-circular path in the i, j -plane from i to $-i$. At the beginning of the path, $C(m) = imi$, while at the end of the path $C(m) = (-i)m(-i) = imi$. Thus C describes a loop in $SO(8)$. At the beginning of the path, $(A(u), B(v)) = (iu, -iv)$, while at the end $(A(u), B(v)) = (-iu, iv)$. Hence (A, B) describes a non-loop in G . Thus G is the non-trivial double covering $\text{Spin}(9)$ of $SO(9)$. QED

Here is a further indication of the extent of symmetry of the Hopf fibration $H: S^7 \hookrightarrow S^{15} \rightarrow S^8$. Orient the fibres.

PROPOSITION 7.10. *Let P and Q be any two fibres of H . Then a preassigned orientation preserving rigid motion of P onto Q can be extended to a symmetry of H . In particular, the symmetries act transitively on S^{15} .*

By Lemma 7.7, the symmetries act transitively on fibres, so we may take $P = Q = L_0$. To preassign an orientation preserving rigid motion of L_0 onto itself is to preassign the map $A \in SO(8)$ in the Triality Principle, which then promises the desired symmetry of H . QED

BIBLIOGRAPHY

- [Ca 1] CARTAN, Élie. Le principe de dualité et la théorie des groupes simples et semi-simples. *Bull. des Sciences Math.* 49 (1925), 361-374.
- [Ca 2] ———. Certaines formes riemanniennes remarquables des géométries à groupe fondamental simple. *Annales École Normale* 44 (1927), 345-467.
- [Cu] CURTIS, C. W. The four and eight square problem and division algebras. In *Studies in modern algebra*, A. Albert ed., MAA (1963), 100-125.
- [Es] ESCOBALES, R. H. Riemannian submersions with totally geodesic fibres. *J. Diff. Geom.* 10 (1975), 253-276.
- [Fr] FREUDENTHAL, Hans. *Oktaven, Ausnahmegruppen und Octavengeometrie*. Utrecht (1951).

- [GWZ] GLUCK, Herman, Frank WARNER and Wolfgang ZILLER. Fibrations of spheres by parallel great spheres and Berger's rigidity theorem. To appear.
- [HL] HARVEY, Reese and H. Blaine LAWSON, Jr. Calibrated geometries. *Acta Math.* 148 (1982), 47-157.
- [Ho 1] HOPF, Heinz. Über die Abbildungen der dreidimensionalen Sphäre auf die Kugelfläche. *Math. Ann.* 104 (1931), 637-665.
- [Ho 2] ——— Über die Abbildungen von Sphären auf Sphären niedrigerer Dimension. *Fund. Math.* 25 (1935), 427-440.
- [Hu 1] HURWITZ, A. Über die Komposition der quadratischer Formen von beliebig vielen Variablen. *Nachrichten der Konigl. Ges. der Wiss. Gottingen* (1898), 309-316.
- [Hu 2] ——— Über die Komposition der quadratischer Formen. *Math. Annalen* 88 (1923), 1-25.
- [Ja] JACOBSON, N. Composition algebras and their automorphisms. *Rend. d. Cir. Mat. di Palermo* 7 (1958), 55-80.
- [Pe] PENROSE, Roger. The geometry of the universe. In *Mathematics today*, Lynn Steen ed., Springer-Verlag (1978), 83-125.
- [Ra] RANJAN, Akhil. Riemannian submersions of spheres with totally geodesic fibres. *Preprint* (1983).
- [Ro] ROBERT, E. *Composition des formes quadratiques de quatre et de huit variables indépendantes*. Thesis, Zürich (1912).
- [Wol 1] WOLF, J. Geodesic spheres in Grassmann manifolds. *Illinois J. Math.* 7 (1963), 425-446.
- [Wol 2] ——— Elliptic spaces in Grassmann manifolds. *Illinois J. Math.* 7 (1963), 447-462.
- [Won] WONG, Y.-C. Isoclinic n -planes in Euclidean $2n$ -space, Clifford parallels in elliptic $(2n-1)$ -space, and the Hurwitz matrix equations. *Mem. Amer. Math. Soc.* 41 (1961).

(Reçu le 14 janvier 1985)

Herman Gluck
Frank Warner
Wolfgang Ziller

Department of Mathematics
University of Pennsylvania
Philadelphia, 19104-6395 (USA)