

2. Fixed points for homeomorphisms of the sphere

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **33 (1987)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

2. FIXED POINTS FOR HOMEOMORPHISMS OF THE SPHERE

The next lemma is the second important ingredient. It can be proved by Nielsen's theory of fixed points. We will give a direct proof.

LEMMA 2.1. *Let $h: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ be an orientation preserving homeomorphism. If h has a period 2 point which is not a fixed point, then the set $\text{Fix}(h)$ can be written as a disjoint union $\text{Fix}(h) = F_1 \cup F_2$ with F_1 and F_2 closed non empty and having point index equal to 1.*

Proof. Call x the point of period 2. Remark that since h preserve the orientation it induces on $\pi_1(\mathbf{S}^2 \setminus \{x, h(x)\}) = \mathbf{Z}$ the map $x \mapsto -x$. Choose an essential annulus $A \subset \mathbf{S}^2 \setminus \{x, h(x)\}$ large enough so that when we compose $h: A \rightarrow \mathbf{S}^2 \setminus \{x, h(x)\}$ with a retraction of $\mathbf{S}^2 \setminus \{x, h(x)\}$ on A we obtain a map $\bar{h}: A \rightarrow A$ which has no fixed point on the boundary, has the same fixed point as h and is equal to h in a neighborhood of the set of fixed points $\text{Fix}(h) = \text{Fix}(\bar{h})$. We will call $\tilde{A} \rightarrow A$ the universal cover of A of course $\tilde{A} = [0, 1] \times \mathbf{R}$ and if we denote by T a generator of the group of deck transformation of $\tilde{A} \rightarrow A$, we can write under this identification $T(x) = x + 1$ where addition is to be taken in the \mathbf{R} coordinate. The map \bar{h} lifts to a proper map \tilde{h} which verifies $\tilde{h}T = T^{-1}\tilde{h}$. It follows that \tilde{h} can be extended to the compactification of \tilde{A} by its two ends $\varepsilon_-, \varepsilon_+$ by a map which exchange these to ends. Since $\tilde{A} \cup \{\varepsilon_-, \varepsilon_+\}$ is homeomorphic to a disk \tilde{h} has a non empty compact set \tilde{F}_1 of fixed points which does not intersect the boundary because \tilde{h} exchange ε_- and ε_+ and \bar{h} has no fixed point on the boundary of A . Remark that the index of \tilde{F}_1 is 1. Moreover, the map $\tilde{A} \rightarrow A$ is injective on \tilde{F}_1 because if $\tilde{h}(x) = x$ we have $\tilde{h}(x+n) = x - n \neq x + n$ if $n \neq 0$. Since $\tilde{A} \rightarrow A$ is a covering it is clear that this map is also injective in a neighborhood of \tilde{F}_1 . It follows that the image F_1 of \tilde{F}_1 under $\tilde{A} \rightarrow A$ is a compact non empty set of fixed points of \bar{h} which has index 1. If $x \in \tilde{F}_1$, we have $T\tilde{h}(x+n) = T(x-n) = x + 1 - n \neq x + n$ for all n because $1/2 \notin \mathbf{Z}$. It follows that F_2 , the image under $\tilde{A} \rightarrow A$ of $\text{Fix}(T\tilde{h})$ —which is also a compact non empty set of fixed points of \bar{h} with index 1—is disjoint from F_1 . If $x \in A$ is a fixed point of \bar{h} , it lifts to a point $\tilde{x} \in \tilde{A}$ which verifies $\tilde{h}(\tilde{x}) = \tilde{x} + n$. If $n = 2k$ then $\tilde{h}(\tilde{x}+k) = \tilde{h}(\tilde{x}) - k = \tilde{x} + 2k - k = \tilde{x} + k$. If $n = 2k - 1$ then $T\tilde{h}(\tilde{x}+k) = T(\tilde{x}+2k-1-k) = \tilde{x} + k$. This shows clearly that $\text{Fix}(\bar{h}) = F_1 \cup F_2$. Since \bar{h} is equal to h in a neighborhood of $\text{Fix}(\bar{h}) = \text{Fix}(h)$, this ends the proof. \square

If we combine the Main Lemma 1.1 and lemma 2.1, we obtain:

LEMMA 2.2. *Let $h: \mathbf{S}^2 \rightarrow \mathbf{S}^2$ be an orientation preserving homeomorphism. If h has a non wandering point which is not a fixed point, then the set $\text{Fix}(h)$ can be written as a disjoint union $\text{Fix}(h) = F_1 \cup F_2$ with F_1 and F_2 closed non empty and having fixed point index equal to 1.*

Since we can compactify an orientation preserving homeomorphism of \mathbf{R}^2 by an orientation preserving homeomorphism of \mathbf{S}^2 with one more fixed point at infinity, we obtain the next two corollaries.

COROLLARY 2.3. (Brouwer's Lemma on translation arcs). *Let $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be a fixed point free orientation preserving homeomorphism. Then h has no periodic point, each point wanders under h . Moreover, if α is a translation arc, the union $\bigcup_{n \in \mathbf{Z}} h^n(\alpha)$ is homeomorphic to a line and it does not accumulate on itself.*

COROLLARY 2.4. *Let $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ be an orientation preserving homeomorphism. If the non wandering set of h is not reduced to the set of fixed points then there is a compact non empty subset $F \subset \text{Fix}(h)$ which has fixed point index equal to 1.*