§3. Harmonic Maps

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3. If M is a compact Riemannian manifold covered conformally by $\Omega \subset S^n$ with $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$, then M admits a metric with scalar curvature $R \geqslant 0$ in the same conformal class. It is conjectured that if $R \geqslant 0$, then $\mu_{\frac{n}{2}-1}(\Lambda) < \infty$.

The basic idea is that by using the developing map, we can reduce the problems to the study of a scalar equation, namely the Yamabe equation on an open subset of S^n. The remaining parts of the proofs are relatively easy. By using the same technique, Schoen and Yau proved that for a compact conformally flat manifold with positive scalar curvature, $\pi_i(M) = 0$ for $2 \leqslant i \leqslant n/2$. Some of their results are also valid for complete manifolds.

§ 3. Harmonic Maps

Harmonic maps are important objects in geometry and analysis. They appear naturally as critical points of an energy functional of the appropriate function space. Harmonic maps reflect a lot about the geometric properties of manifolds.

Given Riemannian manifolds M and N, consider the mapping space $C^r(M, N)$. One problem is to find nice (i.e., canonical) representatives in this space. For a map $f: M \to N$ we define its energy by $E(f) = \int_M |df|^2 dV_M$. A harmonic map is a critical point of this energy. The first question is that of existence, uniqueness and regularity.

1. Existence, uniqueness and regularity

The first major work was done by J. Eells and L. Sampson [ES]. They proved the existence of a harmonic map in any homotopy class in the case where M and N are compact manifolds with $K_N \leqslant 0$. They deformed an arbitrary map through a nonlinear heat equation. By passing to the limit, with the appropriate estimates, one obtains a harmonic map in this way. In fact, harmonic maps are unique in their homotopy classes if $K_N < 0$ and rank $\geqslant 2$ [Hr]. Later, R. Hamilton [Ha] using the same method as in [ES] together with delicate estimates, settled the Dirichlet problem when M

is a manifold with boundary. This type of argument breaks down when we drop the non-positivity condition. For example Eells and Wood [EW1] have shown that there does not exist a degree 1 map from a 2-torus to a 2-sphere.

Instead of looking for harmonic maps in a homotopy class, one can look for harmonic maps with the same action on π_1. We say that two maps $f, g: M \to N$ are π_1- equivalent if $f_* = g_*: \pi_1(M) \to \pi_1(N)$. When M is a Riemann surface, L. Lemaire [Lm] proved the existence of a regular, energy minimizing harmonic map in the class of π_1-equivalent maps.

Another treatment of this problem was given by Sacks-Uhlenbeck [SaU] and R. Schoen-S. T. Yau [Sc-Y1]. Schoen-Yau considered the function space L_1^2 and showed that for $u \in L_1^2(M, N)$, u_* is well-defined and preserved under the weak limit. Using the class $\{f \in L_1^2(M, N) \mid f_* = (f_0)_*\}$ which is weakly closed, combined with the regularity of minimizing harmonic maps from a surface, one can show the existence of a smooth harmonic map in this class.

Schoen-Yau's argument could be generalized to higher dimensions by restricting the map f to the two skeleton of M. (This was also observed by White [Wh].) It is reasonable to expect that one can produce an energy minimizing harmonic map whose action on $\pi_2(M)$ has some resemblance to a given map.

For minimizing harmonic maps, R. Schoen and K. Uhlenbeck [ScU1, 2] have done fundamental work. By delicate use of comparison maps, they showed that the Hausdorff dimension of the singular set of energy minimizing harmonic maps is of codimension at least three. Their theorem can be used to recover the former theorems of Eells-Sampson and Sachs-Uhlenbeck.

2. Noncompact manifolds

The theory for harmonic maps between noncompact manifolds is more complicated than when the manifolds are compact. One reason is that when we choose a minimizing sequence of maps, their energies may not be concentrated in a bounded region. On the other hand, one hopes that this can be prevented by making suitable topological assumptions on the manifolds.

For L^2-harmonic maps, i.e., weakly harmonic maps with finite energy, one can sometimes prove existence by making geometric or topological restrictions. When N is a manifold with nonpositive curvature, Schoen and

Yau [Sc-Y2] have generalized Eells-Sampson's [ES] and Hartman's [Hr] work. They showed that if N is a compact manifold with nonpositive sectional curvature, M is complete and $f: M \to N$ has finite energy, then f is homotopic on compact sets to a harmonic map with finite energy.

Later [Sc-Y3], by explicitly computing the hessian of the distance function d^2 considered as a function on $N \times N$, showed that the set of harmonic maps in a homotopy class is connected (see [Hr] when M is compact) and can be immersed in N as a totally geodesic submanifold. Moreover, it is a point if $\pi_1(N)$ has no nontrivial abelian subgroup and the image of M is neither a point nor a circle. Here we assumed M has finite volume and the harmonic maps have finite energy. (When N is locally symmetric, this is also done by Sunada.) They also applied the theory of harmonic maps to study finite groups acting on a compact manifold.

3. RIGIDITY

It is natural to ask if harmonic homotopy equivalences are isometries when M and N are both negatively curved Einstein manifolds with dimension $\geqslant 3$. This is based on the uniqueness of harmonic maps into negatively curved manifolds and the Mostow rigidity theorem. If this is true, it would give another proof of the Mostow rigidity theorem in the case of rank one symmetric spaces.

It is a question for negatively curved manifolds M and N, whether a harmonic homotopy equivalence is a diffeomorphism or not. Schoen-Yau [Sc-Y4] and Sampson [Sa] have proved this when M and N are Riemann surfaces. If we only assume non-positivity of curvature, Calabi has constructed a counterexample when N is a torus.

By minimizing the energy among diffeomorphisms, combined with a replacement argument, Jost-Schoen [JS] constructed a harmonic diffeomorphism between surfaces of the same genus without any curvature assumption. (Hence it generalizes a theorem of Schoen-Yau where one assumes the image has non-positive curvature.)

There are plenty of examples of harmonic maps when M and N are Kähler manifolds. In particular, holomorphic maps are harmonic. On the other hand, it was conjectured by Yau that when N has negative curvature, harmonic maps are holomorphic. In attempting to settle this conjecture of Yau, Siu [S2], proved that a harmonic map f is either holomorphic or antiholomorphic provided N is strongly negatively curved and the rank of f is not less than 4 at some point. The assumption of N being strongly negatively curved is similar to the negativity of the curvature operator. One expects to be able to weaken this condition. But, if one only assumes negative bisectional curvature, the analog of Siu's theorem is false. This is because for $M = B^n/\Gamma$ embedded in $\mathbf{CP}^n$ as a regular subvariety, any hyperplane section of M has negative bisectional curvature and it is not rigid in general.

Recently, Jost-Yau [JY1, 2] looked at the complex structure of complex surfaces M homotopy equivalent to $N = D \times D/\Gamma$ where Γ is irreducible. Let $f: M \to N$ be a harmonic homotopy equivalence where M is Kähler. By analyzing the foliation $f^\alpha \equiv$ const., they showed that the universal cover of M is biholomorphic to $D \times D$.

Subsequently, Mok [Mk2] generalized the theorem of Jost-Yau to arbitrary dimension. He also considered the foliation studied by Jost and Yau.

A generalization of the rigidity theorem to quasi-projective manifolds was made by Jost-Yau. They study the complex structure over Hermitian symmetric spaces with finite volume.

For a compact manifold M with strongly nonpositive curvature, one likes to prove M is either locally Hermitian symmetric or that the complex structure is rigid. Sampson [Sa] treated the case where M is Kähler and N is a Riemannian manifold with Hermitian negative curvature, that is $R^N_{ijkl}\, u^i v^j \bar{u}^k \bar{v}^l \leqslant 0$. By applying Bochner's technique in essentially the same way as Siu, he showed that all harmonic maps between M and N are holomorphic. Using Sampson's result, combined with the existence theorem for harmonic maps, we can easily obtain restrictions on the fundamental group of a Kähler manifold.

Another interesting situation is when M and N are Kähler manifolds and N has positive sectional curvature. Is it true that any minimizing harmonic map is holomorphic or antiholomorphic? This is only known when $M = \mathbf{CP}^1$. Also, if we can prove this assuming in addition that N is an irreducible symmetric space, then the conjecture that an irreducible symmetric Kähler manifold has only one Kähler structure is probably true. Notice that for the reducible Kähler manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$, there exists infinitely many complex structures which are Kähler.

4. Harmonic maps in physics

The classification theory of harmonic maps from surfaces to Riemannian manifolds, especially symmetric spaces, is of interest to mathematical physicists. The simplest symmetric spaces are the real and complex projective spaces. In [Ca1], Calabi gave an effective parametrization of isotropic harmonic maps from surfaces into real projective space. Following Calabi and the work of physicists, Eells and Wood [EW2] set up a bijective correspondence between full isotropic harmonic maps $\phi: M^2 \to \mathbf{CP}^n$ and pairs (f, r) where $f: M^2 \to \mathbf{CP}^n$ is a full holomorphic map and $0 \leqslant r \leqslant n$ is an integer (see [Ca1] and [EW2] for definitions). Their idea is based on the fact that if $\phi: M \to \mathbf{CP}^n$ is a full isotropic map, then for some $r, s, r+s=n$, the map

$$f = [(\phi \oplus D''\phi \oplus \cdots \oplus (D'')^{r-1}\phi \oplus (D'\phi \oplus \cdots \oplus (D')^s\phi)]^\perp$$

is full holomorphic. Here D' and D'' are the $(1,0)$ and $(0,1)$ components of the covariant derivative.

Later, Bryant ([Br1], [Br2]) treated conformal harmonic maps from surfaces into S^6 and S^4. Inspired by the twistor construction of Calabi and Penrose, he considered a restricted class of conformal harmonic maps, namely superminimal surfaces. (Note that Hopf already studied these surfaces in its primitive form). He established a one-to-one correspondence between superminimal surfaces and curves horizontal in $\mathbf{CP}^3$ with respect to the twistor fibration $\mathbf{CP}^3 \xrightarrow{T} S^4$. By constructing such a curve, Bryant showed that any Riemann surface be conformally immersed as a minimal surface in S^4. For the construction in a general 4-manifold, see [ESa].

Recently, K. Uhlenbeck [U3] has dealt with the space H of harmonic maps from a simply-connected 2-dimensional domain into a real Lie group $G_{\mathbf{R}}$ (which is the chiral model in the language of theoretical physics). She studied the algebraic structure of the manifold H and its relation with Kac-Moody algebras.

Another uncultivated area in harmonic maps is the classification of harmonic maps from a surface into a Ricci flat Kähler three-fold. The interest in this comes from the study of superstring theory in theoretical physics.

§ 4. Minimal submanifolds

The study of minimal submanifolds is another important topic in differential geometry. In this section we will mainly consider minimal surfaces in compact three manifolds. The minimal surfaces will be assumed to be regular and embedded, except when otherwise indicated.