6. Ricci Flat Metrics on Noncompact Manifolds

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two, this is well understood. For example, suppose $C \subseteq \bar{M}^2$ is an elliptic curve and $C \cdot C < 0$. If s is a section of the bundle $[C]$ and $C = \{s=0\}$ then $dv_{\bar{M}}/|s|^2(\log|s|^2)^3$ is a complete asymptotic Kähler-Einstein metric on $\bar{M}/C$ with C as the cusps of the metric.

Suppose that D is a divisor on a compact Kähler manifold M satisfying $c_1(K+[D]) \geqslant 0$ on $\bar{M}$, $c_1(K+[D]) > 0$ on $\bar{M}\backslash D$ and $(K+[D]) - \varepsilon[D]|_D > 0$ then $\bar{M}\backslash D$ admits a Kähler-Einstein metric with finite volume. Moreover, the curvature of the metric and all of its covariant derivatives are bounded. It is not clear whether complete Kähler-Einstein metrics should have bounded curvature.

For a quasi-projective manifold $M = \bar{M}\backslash D$, a Kähler-Einstein metric always has finite volume and one can define logarithmic Chern classes $\tilde{c}_i(M, D)$. The existence of the Kähler-Einstein metric implies the following inequality for the log Chern classes $\tilde{c}_1$ and $\tilde{c}_2$:

$$(*') \qquad (-1)^n \tilde{c}_1^{n-2} \cdot \tilde{c}_2 \geqslant \frac{(-1)^n}{2(n+1)} \tilde{c}_1^n .$$

A particularly significant fact is that equality holds in (*) if the quasi-projective manifold $\bar{M}\backslash D$ is the quotient of the unit ball in $\mathbf{C}^n$.

Recall that a complex manifold is called measure hyperbolic if the Kobayashi measure is positive everywhere. Moreover, for a complete Kähler-Einstein manifold, the following inequality holds,

$$c_1 dv_{\text{Kobayashi}} \geqslant dv_{\text{Kähler-Einstein}} \geqslant c_2 dv_{\text{Caratheodory}}$$

where c_1 and c_2 are two universal positive constants. We have the following question: If the Caratheodory metric of M is complete, does M admit a complete Kähler-Einstein metric?

6. Ricci Flat Metrics on Noncompact Manifolds

We now consider Ricci flat metrics on a complete, noncompact manifold M. We first remark that in this case uniqueness is unknown. Even for compact manifolds, Kähler-Einstein metrics are only unique in each Kähler class. Suppose g and g' are two Ricci flat Kähler metrics on M. If they satisfy $g_{i\bar{j}} - g'_{i\bar{j}} = \partial\bar{\partial}F$ with F bounded, then $g_{i\bar{j}} = g'_{i\bar{j}}$. Note that in the compact case, the above condition means that g and g' belong to the same Kähler class. It also may be possible to drop the condition that F is bounded since there do not exist too many Ricci flat metrics.

In any case, the uniqueness problem is far from solved. Even when $M = \mathbf{C}^n$, Calabi proposed the following open problem: If $u: \mathbf{C}^n \to R$ is a strictly plurisubharmonic function with $\det\left(\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}\right) = 1$, then if the Kähler metric $ds_u^2 = \sum \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} dz^i \otimes d\bar{z}^j$ is complete does it have zero curvature? Note that ds_u^2 is not complete in general. For example, Fatou and Bieberbach (see the book of Bochner and Martin [B-M], p. 45) gave a biholomorphism $F: \mathbf{C}^2 \to \Omega$, where $\Omega \subseteq \mathbf{C}^2$ is open and $\mathbf{C}^2/\Omega$ contains an open set, such that the Jacobian of F is identically equal to one. For $u = |z^1|^2 + |z^2|^2$, $ds^2_{u\,.\,F} = F^*ds_u^2 = F^*ds_0^2$ is not complete.

There are a lot of biholomorphisms F in Aut$(\mathbf{C}^2)$ with Jacobian equal to one; for example, let $F(z, w) = (z + f(w), w)$ for any entire function f. For the above u, $u \circ F$ is still strictly plurisubharmonic and $ds_{u\,.\,F}$ is complete and Ricci flat. Thus, intuitively, the larger the group Aut(M), the more difficult the problem is.

We now consider the question of existence. Just as in the case of negative scalar curvature, the existence of a complete, Ricci flat, Kähler metric will impose restrictions on the complex structure of M. For example, by the Schwarz lemma [Y4], we know that there does not exist any nontrivial holomorphic maps from M to a Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant. As a corollary, if there exists a nontrivial holomorphic map from $\bar{M}$ to an algebraic curve of genus greater than one, then $M \subseteq \bar{M}$ cannot admit any complete Kähler metric with nonnegative Ricci curvature.

We conjecture that if M^n admits a complete Ricci flat Kähler metric, then $M = \bar{M}\backslash$(divisor) where $\bar{M}$ is compact and Kähler. This would mean that the infinity of M cannot be too large. Now suppose $M^2 = \bar{M}\backslash$(divisor) and dv is a Ricci flat volume form on M. One would like to determine M; by going to the universal cover, we can assume M is simply connected. Locally, $dv = (\sqrt{-1})^2 k dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2$ for some positive real function k. Since Ric$(dv) = 0$, we have $\partial\bar{\partial}(\log k) = 0$ and k can be written as $k = |h|^2$ for some locally defined holomorphic function h. By a monodromy argument, we obtain a holomorphic 2-form $\eta = h dz^1 \wedge dz^2$, with h nowhere zero and $\eta \wedge \bar{\eta} = dv$. Hence $\eta^{-1} = h^{-1} dz^1 \wedge dz^2$ can be considered as a global section of the anti-canonical bundle K^{-1}.

Intuitively, one might expect that h approaches ∞ near the infinity of M and η^{-1} can be extended to $\bar{M}$, that is, there exists a nontrivial section

$S \in H^0(\bar{M}, K^{-1})$. This would imply that either K is trivial on M or $H^0(\bar{M}, K^n) = 0$ for every $m > 0$ and hence the Kodaira dimension of $\bar{M}^2$ would either be $-\infty$ or 0. This is because if $t \in H^0(\bar{M}, K^n)$, then $t \cdot S^n$ is a holomorphic function on M and hence constant; since S is zero somewhere unless K is trivial, we have $t \cdot S^n = 0$, so that $t = 0$ unless K is trivial on M.

Since M is Kähler and simply connected, the minimal model of $\bar{M}$ is a Kähler surface with $K = 0$ or $-\infty$ and $b_1 = 0$. When $K = 0$, it is either a $K-3$ surface or Enriques' surface. When $K = -\infty$ it is either a rational surface or a ruled surface of genus zero, $\bar{M}^2$ is equal the minimal model blown up successively at a finite number of points, and $M = \bar{M} \backslash \{s=0\}$ for some $0 \neq s \in H^0(\bar{M}, K^{-1})$. Conversely, if $M = \bar{M} \backslash \{s=0\}$ with $s \in H^0(\bar{M}, K^{-1})$ and $\bar{M}$ is as above, then M should admit a Ricci flat, complete, Kähler metric. In higher dimensions, the situation is much more complicated.

In physics, the following question has been studied. Is a Ricci flat metric with a suitable locally asymptotic property actually unique? This is the case when the metric is asymptotically flat. One would also like to know what happens when the metric is locally asymptotic to a cone. Perhaps assuming that the metric is Kähler may make this problem easier.

The existence of Ricci flat metrics has many applications. For example, using Ricci flat metrics, Siu [S1] proved that any surface M^2 with $c_1(M) = 0$ and $H^1(M, \mathbf{R}) = 0$ must be Kähler. See also Todorov [To] for higher dimensions. One can also ask the following question: Let M^{2n} be a simply-connected, compact, complex manifold where $n \geqslant 2$. If there exists a non-degenerate 2-form $\omega \in H^{2,0}(M)$, is M then Kähler? Todorov claimed that M is Kähler under an additional assumption: $\dim H^{2,0}(M) = 1$.

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