§5. Proof of Theorem 2

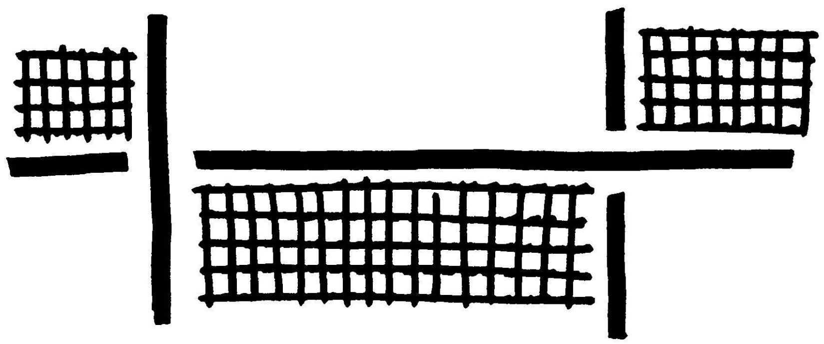

FIGURE 19

Observe that two unshaded regions near one crossing point are necessarily distinct, otherwise the diagram K would not be reduced:

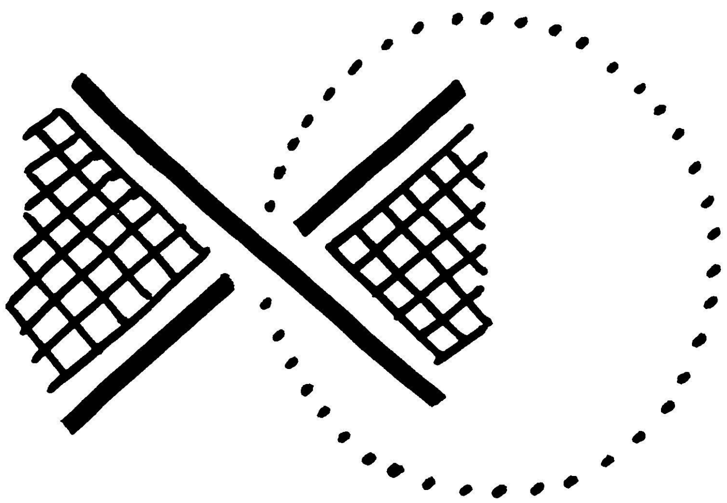

FIGURE 20

It is evident that A is equal to the number of unshaded regions. Let a state S^2 be obtained from A by replacing one positive marker by the negative marker. Under this operation two distinct unshaded regions are connected by a band, and therefore $|S^2| = |A| - 1$. In view of the arguments given in the proof of part (i) of the Theorem, this implies that $D_S < D_A$ for any state S of K. This implies (8). Analogous arguments imply (9), and the proof of (ii) in Theorem 1 is complete.

§ 5. PROOF OF THEOREM 2

Let me first recall the definition of the *signature* of an oriented link L in terms of a (not necessarily orientable) surface V bounded by L (see [2]). One defines a bilinear form

$$Q = Q_V : H_1(V; Z) \times H_1(V; Z) \to Z$$

as follows. Let $\alpha, \beta \in H_1(V; Z)$ be represented by loops a, b in V. Let us double all points of a and push them in $S^3 - V$ along both normal directions to V, at the same small distance. We obtain an oriented closed 1-manifold $\tilde{a} \in S^3 - V$; the following picture shows the local situation. The natural projection $\tilde{a} \to a$ is of course a 2-sheeted covering.

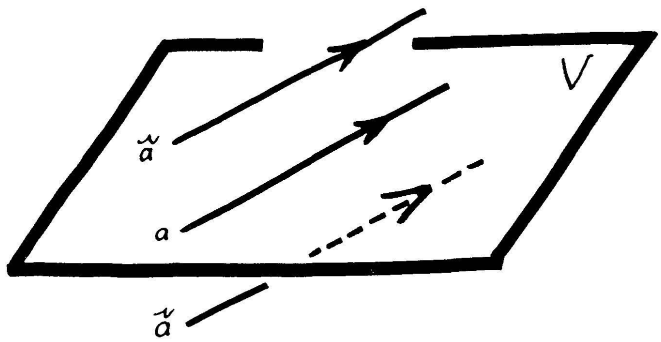

FIGURE 21

Denote by $Q(\alpha, \beta)$ the linking coefficient $Lk(\tilde{a}, b)$ of $\tilde{a}$ and b. It turns out that Q is a well defined symmetric bilinear form. Let L^V be a parallel copy of L in $S^3 - V$. Define

$$\sigma(L) = \text{sign}\,(Q) - \frac{1}{2} Lk(L, L^V).$$

Here sign (Q) denotes the signature of the symmetric bilinear form obtained by factorizing out the annihilator of Q. According to [2], $\sigma(L)$ does not depend on the choice of the spanning surface V. In case V is orientable, $Lk(L, L^V) = 0$ and we get the classical definition of the signature of L due to Murasugi.

All diagrams and links being oriented, it is easy to check that the writhe number of a link diagram, the signature of a link, and the number $d_{\max}(V_L(t)) + d_{\min}(V_L(t))$ are additive with respect to both disjoint unions and connected sums of diagrams. Therefore it is enough to prove Theorem 2 for a diagram K which is connected, prime, alternating and reduced.

Let c_+ and c_- denote the numbers of positive and negative crossing points of such a K.

CLAIM (Murasugi). *One has* $\sigma(L) = |A| - 1 - c_+$.

This claim implies Theorem 2. Indeed, formulas (8), (9) and (6) show that

$$d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K)$$
$$= -w(K)/2 + D_A + d_B = -w(K)/2 + (|A|-|B|)/2\,.$$

Substituting in the last expression

$$w(K) = c_+ - c_-$$
$$|B| = c + 2 - |A|$$
$$c = c_+ + c_-$$

we obtain

$$d_{\max}(V_L(t)) + d_{\min}(V_L(t)) + w(K)$$
$$= |A| - 1 - c_+ = \sigma(L)\,.$$

This implies Theorem 2.

Proof of the Claim. There is a spanning surface V of L associated with the diagram K. It is built up from shaded regions of $S^2 - K$ (see § 4) and small bands connecting these regions which enter one crossing point. In a neighbourhood of a crossing point, V looks like this:

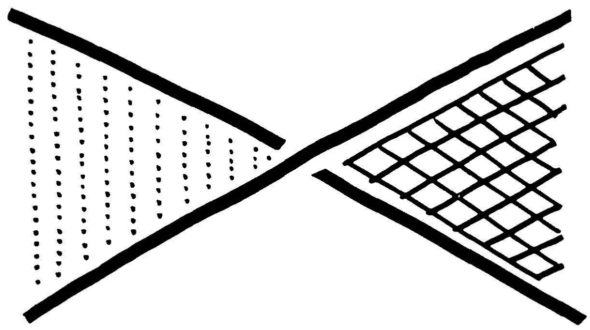

FIGURE 22

We shall prove the claim by using this surface V.

We prove first that the number $-\dfrac{1}{2} Lk(L, L^V)$ is equal to $-c_+$. We may assume that the push-off L^V of L in $S^3 - V$ lies in the unshaded regions of R^2 except in a neighbourhood of the crossing points. The following picture shows L^V near a crossing point (the orientations of L and L^V are not shown).

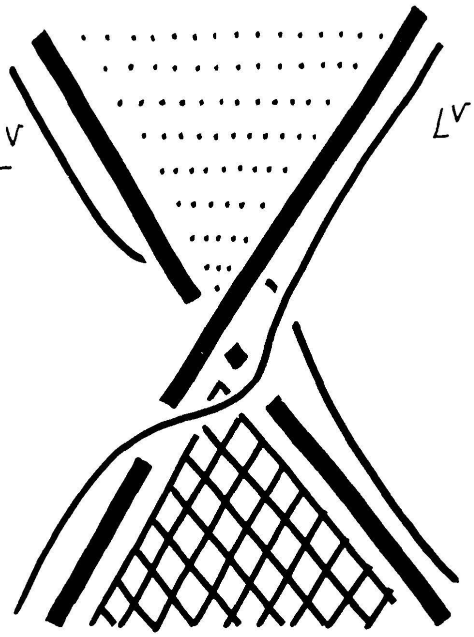

FIGURE 23

We compute $Lk(L, L^V)$, by counting the algebraic number of times L^V passes under L. It is easy to check that each crossing point of L contributes with a 2 if it is positive and with a 0 if it is negative. Thus $Lk(L, L^V) = 2c_+$.

Now, we prove that $\operatorname{sign}(Q_V) = |A| - 1$. The surface V retracts by deformation onto the complement on the unshaded regions in S^2. As the diagram is alternating, the number of unshaded regions is $|A|$, so that $b_1(V) = |A| - 1$. Thus we have to prove that the form Q_V is positive definite.

Let $\alpha \in H_1(V; Z)$ and let a be an oriented closed 1-manifold (possibly non connected) in V which represents α. Thus $Q(\alpha, \alpha) = Lk(\tilde{a}, a)$, where $\tilde{a}$ is the oriented closed 1-manifold in $S^3 - V$ obtained from a by the 2-sheeted blowing up procedure. If a subarc x of a lies in a shaded region far from crossing points of K, then, of the two corresponding subarcs of $\tilde{a}$, one lies over R^2 and the other one lies under R^2. We shall always picture the first (higher) subarc of $\tilde{a}$ on the right side of x (looking from above along a) and the second (lower) subarc of $\tilde{a}$ on the left side of x; see the following picture.

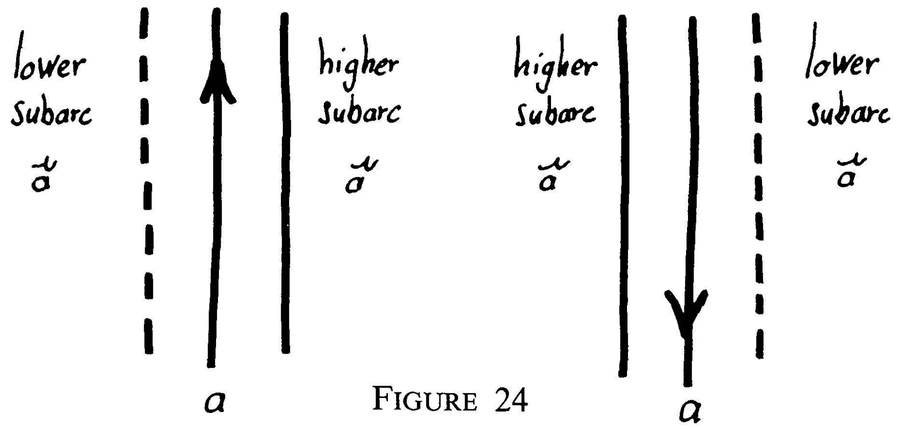

FIGURE 24

Note that the diagram of $\tilde{a}$ misses the diagram of a except in a neighborhood of the crossing points. Surgering if necessary a in V, we may assume that all components of a go through any band of V in one direction. Positions of a like those in the following picture may easily be removed by surgery.

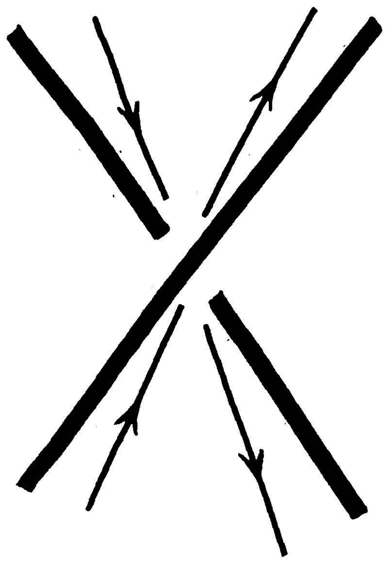

FIGURE 25

For simplicity, consider first a neighbourhood of a crossing point through which a goes only once:

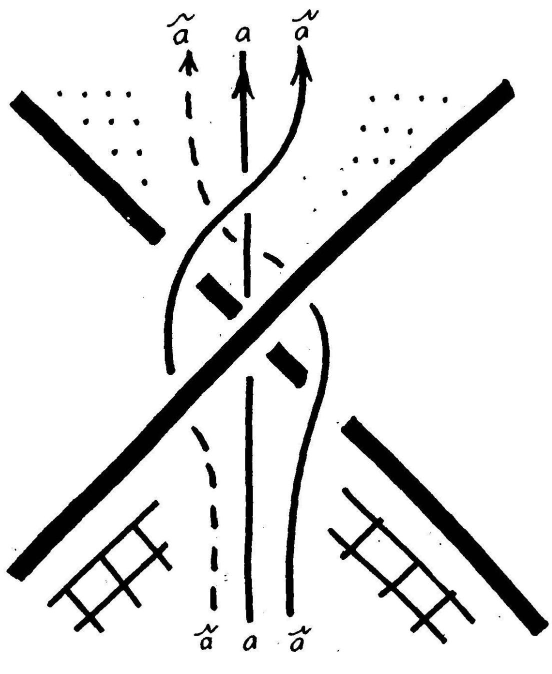

FIGURE 26

It is clear that $\tilde{a}$ passes under a in this neighbourhood one time from right to left.

If a goes through a neighbourhood $\mathcal{U}$ of a crossing point n times, then the relative positions of the corresponding n arcs of a, say $x_1, \ldots, x_n$, are represented as follows:

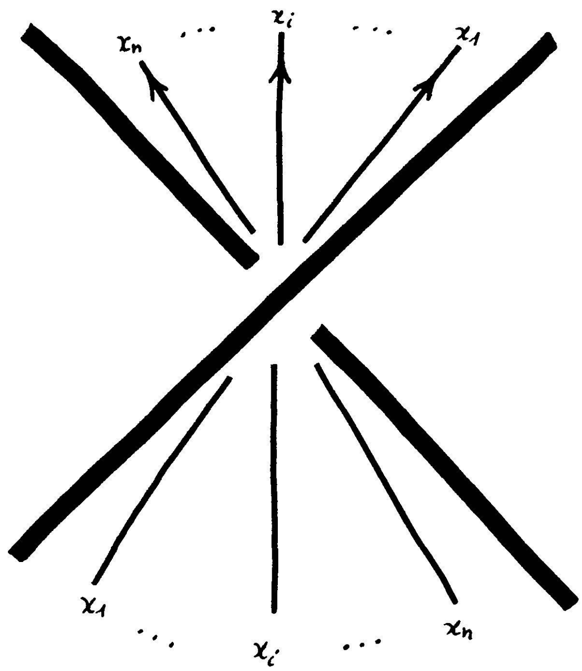

FIGURE 27

In the next picture, we show the two arcs of $\tilde{a}$ which correspond to x_i:

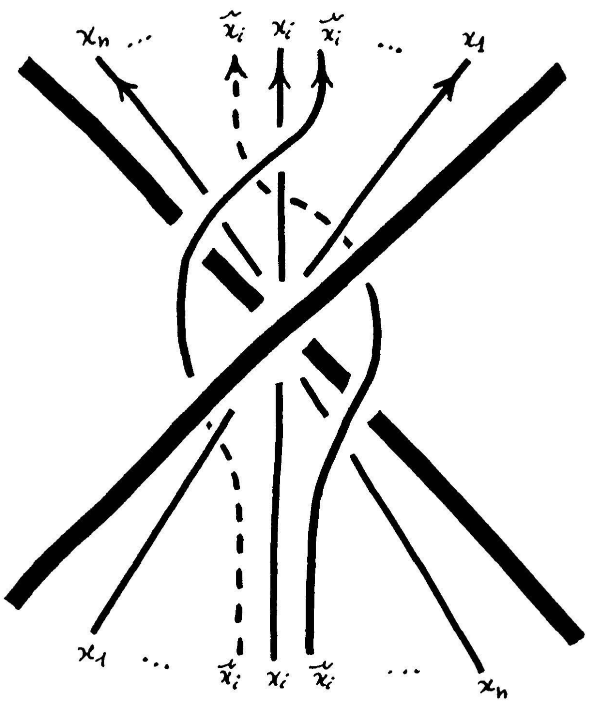

FIGURE 28

It is clear that these two arcs of $\tilde{a}$ pass $2i - 1$ times from right to left under a. Thus the contribution of the neighbourhood $\mathcal{U}$ to $Q(\alpha, \alpha)$ is given by

$$\sum_{i=1}^{n} (2i-1) = -n + 2 \sum_{i=1}^{n} i = n^2 .$$

This shows that $Q(\alpha, \alpha) > 0$ if a crosses at least one band of V. If not, then $\alpha = 0$.

Thus Q is positive definite. This completes the proof of Theorem 2.

Appendix: an improvement of the inequality of Theorem 1

Though the inequality

(10) $$c(K) + r(K) - 1 \geqslant \text{span}\,(L)$$

of Theorem 1 becomes an equality for weakly alternating diagrams, it may be sharpened a little for other cases. Let K be a link diagram in R^2 and let $\Gamma \subset R^2$ be the associated link projection. For $P \in S^2 - \Gamma$ (where $S^2 = R^2 \cup \{\infty\}$), let $i(P)$ be the intersection number modulo 2 of Γ with a generic 1-chain connecting P to ∞. Shade the regions of $S^2 - \Gamma$ for which $i \equiv 1 \pmod 2$, so that S^2 is painted like a chessboard. Let $b_1, ..., b_m$ be the shaded regions of $S^2 - \Gamma$ and let $w_1, ..., w_n$ be the unshaded regions of $S^2 - \Gamma$.

An edge e of Γ is called *K-good* either if e is a loop or if one of the end points of e corresponds to an overcrossing point of K and the other end point of e corresponds to an undercrossing point of K. An edge of Γ which is not K-good is called *K-bad.* For any $i \in \{1, ..., m\}$ and for any $j \in \{1, ..., n\}$, it is clear that the set $\overline{b_i} \cap \overline{w_j}$ consists of several edges and double points of Γ. Denote by $a(i, j)$ the number modulo 2 of K-bad edges in $\overline{b_i} \cap \overline{w_j}$. Denote by $u(K)$ the rank of the $m-by-n$ matrix $\big(a(i, j)\big)$.

THEOREM. *If K is a diagram of a link L, then*

(11) $$c(K) + r(K) - 1 \geqslant \text{span}\,(L) + u(K) .$$

COROLLARY. *If K is a diagram of an unsplittable link L, then*

$$c(K) \geqslant \text{span}\,(L) + u(K) .$$

Of course, if K is a weakly alternating diagram, then $u(K) = 0$.