V. Interlude

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with minimum condition (see sec. VI below). What it left open is important questions about the structure of nilpotent algebras and division algebras (see secs. V, VII). It is perhaps fitting to conclude our discussion of Wedderburn's work with several tributes:

> [Wedderburn] was the first to find the real significance and meaning of the structure of a simple algebra. This extraordinary result has excited the fantasy of every algebraist and still does so in our day. (Artin, [6].)

> Wedderburn's pioneering work on the structure of simple algebras set the stage for the deep investigations—often with an eye to applications in algebraic number theory—in the theory of algebras. (Herstein, [43].)

> [Wedderburn's] rational methods struck at the heart of the theory of algebras, and their influence is felt even to this day... His work neatly and brilliantly placed the theory of algebras in the proper, or at least in the modern, perspective. (Parshall, [66].)

V. Interlude

The title is not meant to suggest a lack of activity in the study of algebras during the first two decades or so of the 20th century. There were simply no *fundamental* developments in the period between the work of Wedderburn in 1907 and the works of Artin, Noether *et al.* in the 1920s. Below we briefly describe two areas of progress in these intervening years.

(a) Division algebras

As we noted, Wedderburn's structure theorems left unresolved the nature of division algebras. Knowledge of finite-dimensional division algebras over a field F was available only in the following three cases:

(i) $F = \mathbf{R}$. In this case, as we have seen, there are only three division algebras over F, namely the reals, complex numbers, and quaternions.

(ii) $F =$ an algebraically closed field (eg. $\mathbf{C}$). In this case Wedderburn himself showed (in the 1907 paper where his structure theorem appears) that over such a field there are no division algebras except for the field itself. As Wedderburn put it:

> If the given field is so extended that every equation is soluble, the only primitive [division] algebra in the extended field is the algebra of one unit, $e = e^2$.

(iii) F = a finite field. Since the algebra is finite-dimensional it must, in fact, be finite in this case. Wedderburn had earlier (1905) established the remarkable result that in such a case the division algebra is commutative (see [81]).[1] The division algebra over the finite field F is thus itself a finite field. Still earlier (1893) E.H. Moore had characterized all finite fields. This case, then, was also completely solved.

No other examples of division algebras were known at that time (in 1905). Thus the only "genuine" (i.e. noncommutative) division algebra known was the quaternions. Dickson noted that the discovery and classification of division algebras was the chief outstanding problem in the theory of algebras over an arbitrary field. He then proceeded (in 1905 and 1914) to contribute to its solution by exhibiting new classes of division algebras over an arbitrary field (see [26] for details). He also showed that there are infinitely many nonisomorphic quaternion algebras over the rationals (see [4]). In the 1920s Dickson (and others) defined the important class of *cyclic* division algebras. (See footnote on p. 257 for a definition.) The major step in the study of division algebras was the description, in the early 1930s, of all division algebras over the field of *rational* numbers (see sec. VII).

(b) Definitions of an abstract algebra and an abstract ring

The definition of an "associative algebra" (hypercomplex number system) throughout the 19th century, and even in the early 20th century (for example, in Wedderburn's 1907 paper) was that of a system of elements of the form $\sum a_i e_i$ (a_i elements of a field, e_i "basis" elements), with componentwise addition and with multiplication of the basis elements given by "structural constants", obeying certain laws (see p. 238). In 1903 Dickson gave the first more or less abstract definition of an associative algebra [25]. To him, it is a

> System of elements $A = (a_1, a_2, \ldots, a_n)$ each uniquely defined by n marks of the field F together with their sequence. The marks $a_1, \ldots, a_n$ are called the *coordinates* of A. The element $(0, 0, \ldots, 0)$ is called zero and designated 0. Addition of elements is defined thus:
> $A + B = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n)$.
> It follows that there is an element $D = (a_1-b_1, a_2-b_2, \ldots, a_n-b_n)$ such that $D + B = A$.
> Consider a second rule of combination of the elements having the properties:

[1]) The theorem, for example, provides the only known proof that in a finite projective plane Desargues' theorem implies Pappus' theorem. Artin [6] claims that this theorem of Wedderburn "has fascinated most algebraists to a very high degree."

1. For any two elements A and B of the system, $A \cdot B$ is an element of the system whose coordinates are bilinear functions of the coordinates of A and B, with fixed coefficients belonging to F.
2. $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, if $A \cdot B, B \cdot C, (A \cdot B) \cdot C, A \cdot (B \cdot C)$ belong to the system.
3. There exists in the system an element I such that $A \cdot I = A$ for every element A of the system.
4. There exists in the system at least one element A such that $A \cdot Z \neq 0$ for any element $Z \neq 0$.

Dickson then goes on to show that "any system of elements given by [this] definition is a system of complex numbers according to the usual ... definition." This he does by first *proving* the distributive law and the uniqueness of the identity element I. Dickson then proves the independence of these postulates.

This "abstract" definition of an associative algebra (a coordinate-free, entirely abstract definition of an algebra was given by Dickson in 1923—see [28]) was one instance of a general interest by American mathematicians around this time in abstract, postulational definitions of algebraic systems and, in particular, in establishing the independence of the postulates of such systems. For example, definitions of groups were given (between 1901 and 1905) by Huntington, E.H. Moore, Dickson, and Pierpont, and of fields (in 1903) by Dickson and Huntington. These definitions of groups and fields were entirely abstract (even from our point of view).[1]) See [8], [10], [12], [49].

In 1914, Fraenkel, in a paper entitled "On zero divisors and the decomposition of rings" [31], was the first to give an abstract definition of a *ring* (cf. p. 2). He gave diverse examples of the concept he was defining, which included both commutative and non-commutative rings, namely integers modulo n, hypercomplex number systems, matrices, and p-adic integers. It was an abstract definition in today's style. Thus Fraenkel defines a ring as "a system" with two (abstract) operations, to which he gives the names addition and multiplication. Under one of the operations (addition) the system forms a group (he gives its axioms). The second operation (multiplication) is associative and distributes over the first operation. Two axioms give the closure of the system under the operations, and there is the requirement of an identity in the definition of the ring. Commutativity under addition does *not* appear as an axiom but is proved!; so are other elementary properties of

[1]) Somewhat earlier (at the end of the 19th Century) we witness the emergence of an abstract, axiomatic approach in geometry (Pasch, Peano, Hilbert) and arithmetic (Dedekind, Frege, Peano).

a ring such as $a \times 0 = 0$, and $a(-b) = (-a)b = -(ab)$. There are two extraneous axioms (dealing with "regular" elements in the ring) which depart from an otherwise modern definition.

Among the main concepts introduced are "zero divisors" and "regular elements". Fraenkel deals in this paper only with rings which are not integral domains and discusses divisibility for such rings. Much of the paper deals with decomposition of rings as direct products of "simple" rings (not the usual notion of simplicity).

Fraenkel's aim in this paper was to do for rings what Steinitz had just (1910) done for fields, namely to give an abstract and comprehensive theory of (commutative and noncommutative) rings.[1]) Of course he was not successful (he does admit that the task here is not as "easy" as in the case of fields)—it was too ambitious an undertaking to try to subsume the structure of both commutative and noncommutative rings under one theory. Fraenkel did, however, delineate the abstract notion of a ring and, in this respect, made a significant contribution.

VI. Structure of rings with minimum condition

In a fundamental paper of 1927 entitled "Zur Theorie der hyperkomplexen Zahlen" [5], Artin proved a structure theorem for rings with minimum condition (descending chain condition)[2]) which generalized Wedderburn's structure theorem for finite-dimensional algebras (discussed in sec. IV). The theorem, now known as the Wedderburn-Artin theorem for semi-simple rings with minimum condition (i.e. rings without nilpotent ideals and satisfying the descending chain condition for, say, right ideals—see e.g. [43]) states that if R is such a ring, then it is a direct sum of simple rings and these, in turn, are matrix rings over division rings; moreover, the above representations are unique (cf. Wedderburn's structure theorem, p. 246).

As we note, the *result* is essentially the same as Wedderburn's. It is, however, the spirit of the work and the conceptual advances which make it

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[1]) Steinitz' "Algebraische Theorie der Körper" of 1910 was the first abstract study of fields as a distinct subject. This fundamental work, which some say marked the beginning of modern abstract algebra, arose out of a desire to delineate the abstract notions common to the various contemporary theories of fields. It provided the basic concepts of field theory necessary for the subsequent abstract study of Galois theory, algebraic number theory, and algebraic geometry.

[2]) Artin proved his theorem for rings satisfying both the ascending and descending chain conditions. Later (1939) Hopkins showed that the descending chain condition suffices.