2. Biduality

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A smooth map $f : X \to Y$ will preserve the degree of a zero cycle in the sense that

(1.9) $$\mathrm{dg}\,(f_* T) = \mathrm{dg}\, T\,, \quad T \in D_0^c(X, \mathbf{C})\,,$$

as it follows from (1.5).

The reader is invited to replace **C** by **R** and change the meaning of the symbol Ω from complex to real differential forms.

2. Biduality

In this section we shall show that de Rham cohomology can be calculated as the linear dual of de Rham homology in the same way singular cohomology can be obtained from singular homology.

(2.1) Theorem. *Let X denote a smooth manifold. Evaluation of a compact p-chain against a p-form induces an isomorphism*

$$H^p(X, \mathbf{C}) \xrightarrow{\sim} \mathrm{Hom}\,(H_p^c(X, \mathbf{C}), \mathbf{C})$$

for all integers p.

Proof. The hart of the matter is of sheaf theoretic nature, so we start with a brief review during which the reader is invited to change the meaning of the letter X to denote a general locally compact space and the letter **C** to denote an arbitrary field. For notation and details the reader may consult [5] V.1, and the references given there.

To a *soft* **C**-sheaf F on X we can associate the sheaf $F^\vee$ whose sections over the open subset U of X are given by

(2.2) $$\Gamma(U, F^\vee) = \mathrm{Hom}\,(\Gamma_c(U, F), \mathbf{C})$$

Restriction in the sheaf $F^\vee$ from U to a smaller open subset V is the **C**-linear dual of "extension by zero"

$$\Gamma_c(V, F) \to \Gamma_c(U, F)\,, \quad V \subseteq U\,.$$

The presheaf $F^\vee$ we have described is actually a sheaf and indeed a *soft* sheaf. This allows us to iterate the construction and form $F^{\vee\vee}$. We shall construct a natural *biduality morphism* of **C**-sheaves on X

(2.3) $$F \to F^{\vee\vee}\,.$$

To this end consider the tautological evaluation

$$\Gamma(U, F^{\vee}) \times \Gamma_c(U, F) \to \mathbf{C}\,.$$

This can be modified to yield a pairing

$$\mathrm{ev}\colon \Gamma_c(U, F^{\vee}) \times \Gamma(U, F) \to \mathbf{C}$$

namely, for $T \in \Gamma_c(U, F^{\vee})$ and $\omega \in \Gamma(U, F)$ choose $\nu \in \Gamma_c(U, F)$, such that ω and ν has the same restriction to $\mathrm{Supp}(T)$, and put $\mathrm{ev}(T, \omega) = T(\nu)$. The evaluation map may be interpreted as a transformation

$$a_U\colon \Gamma_c(U, F^{\vee}) \to \mathrm{Hom}\big(\Gamma(U, F), \mathbf{C}\big)\,. \tag{2.4}$$

An open subset V of U gives rise to a commutative diagram

$$\begin{array}{ccc} \Gamma_c(V, F^{\vee}) & \overset{a_V}{\to} & \mathrm{Hom}\big(\Gamma(V, F), \mathbf{C}\big) \\ \downarrow & & \downarrow \\ \Gamma_c(U, F^{\vee}) & \overset{a_U}{\to} & \mathrm{Hom}\big(\Gamma(U, F), \mathbf{C}\big) \end{array} \tag{2.5}$$

where the first vertical arrow is "extension by zero" in the soft sheaf $F^{\vee}$ and the second vertical arrow is the linear dual of restriction in the sheaf F. Let us return to the open subset U and consider the composite

$$\Gamma(U, F) \to \mathrm{Hom}\big(\mathrm{Hom}\big(\Gamma(U, F), \mathbf{C}\big), \mathbf{C}\big) \overset{a_U^*}{\to} \Gamma(U, F^{\vee\vee})$$

where the first arrow is the biduality map from linear algebra. By variation of U we obtain the biduality morphism $b\colon F \to F^{\vee\vee}$ announced in (2.3).

Let us now return to the situation at hand and consider the biduality morphism for the de Rham complex.

$$b\colon \Omega^{\bullet} \to \Omega^{\bullet\vee\vee} \tag{2.6}$$

which we shall prove to be a quasi-isomorphism. The question is local, so it suffices to check the case $X = \mathbf{R}^n$, which can be done by the Poincaré Lemma with and without compact support. Both complexes are made of soft sheaves, so we lean on the fact, implicit in the definition of a manifold, that X is countable at infinity to conclude that b induces isomorphisms, compare [5] IV.2.2,

$$H^p(X, \mathbf{C}) \tilde{\to} H^p\Gamma(X, \Omega^{\bullet\vee\vee})\,, \quad p \in \mathbf{N}\,.$$

In order to identify the right hand side notice first that

$$\Gamma(X, \Omega^{\bullet\vee\vee}) = \mathrm{Hom}\big(\Gamma_c(X, \Omega^{\bullet\vee}), \mathbf{C}\big)$$

and second, that the map a_X introduced in (2.4) induces an isomorphism

$$a : \Gamma_c(X, \Omega^{\cdot\vee}) \tilde{\to} D^c_{\cdot}(X, \mathbf{C}) . \tag{2.7}$$

Collect this together to conclude the proof. Q.E.D.

The de Rham homology as defined here agrees with the original theory based on currents [6]: the inclusion of the complex of currents in $\Omega.^\vee$ is a quasi-isomorphism as can be seen by the method used in the last third of the proof above.

As a consequence of the isomorphism (2.7) we can of course redefine de Rham homology as

$$H^c_p(X, \mathbf{C}) = H_p\Gamma_c(X, \Omega^{\cdot\vee}) . \tag{2.8}$$

If the letter c is dropped we obtain Borel Moore homology, compare [5] IX and the references given there.

The biduality theorem 2.1 is certainly related to that of Verdier [7], [1]. In fact most of the material presented here may be extended to a context of similar generality. I hope to return to this point in the near future.

3. Smooth singular homology

Let us consider an n-dimensional smooth manifold X. Integration over smooth singular simplexes defines a map

$$S^\infty_{\cdot}(X, \mathbf{C}) \to D^c_{\cdot}(X, \mathbf{C}) \tag{3.1}$$

from the complex of smooth singular simplexes to the complex of compact chains on X.

(3.2) Theorem. *Integration induces an isomorphism*

$$H^\infty_{\cdot}(X, \mathbf{C}) \tilde{\to} H^c_{\cdot}(X, \mathbf{C})$$

from smooth singular homology to de Rham homology.

Proof. Let us first discuss *Mayer-Vietoris* sequences in de Rham homology. For open subsets U and V of X a Mayer-Vietoris sequence arises from the following exact sequence of complexes

$$0 \to \Gamma_c(U \cap V, \Omega^{\cdot\vee}) \xrightarrow{+} \Gamma_c(U, \Omega^{\cdot\vee}) \oplus \Gamma_c(V, \Omega^{\cdot\vee}) \xrightarrow{-} \Gamma_c(U \cup V, \Omega^{\cdot\vee}) \to 0 . \tag{3.3}$$