

§2. Local representation masses and \mathbb{Z}_p -equivalence OF FORMS

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possible anymore, as the following example shows: By Proposition 1.1 the diagonal forms $f = \langle 1, 2, 2, 2, 4 \rangle$, $g = \langle 1, 1, 2, 4, 4 \rangle$ have the same Gauss sums $\theta(\cdot, f, 2^t) = \theta(\cdot, g, 2^t)$ for all $t \geq 1$, however they are obviously not \mathbf{Z}_2 -equivalent.

The theory of Minkowski reproduced in this section was extended by O'Meara to integral quadratic forms over local fields.

§ 2. LOCAL REPRESENTATION MASSES AND \mathbf{Z}_p -EQUIVALENCE OF FORMS

We identify \mathbf{Q}_p with its topological dual by defining $\langle n, m \rangle = \chi_p(nm)$, where χ_p is Tate's character:

$$\chi_p(a) = \exp\left(2\pi i \sum_{s < 0} a_s p^s\right),$$

if $a = \sum_{s \geq s_0} a_s p^s$. Let dn be the Haar measure of \mathbf{Q}_p normalized by $dn(\mathbf{Z}_p) = 1$. As is well-known, dn is selfdual. Let dx be the Haar measure of \mathbf{Q}_p naturally induced by dn .

Let f be a non-singular integral p -adic quadratic form in $k \geq 1$ variables. We shall deal in this section with the representation mass function given by (0.1) for $\phi = 1_{(\mathbf{Z}_p)^k}$. That is, we define for all $n_o \in \mathbf{Q}_p$:

$$r(n_o, f, \mathbf{Z}_p) = \lim_{U \rightarrow \{n_o\}} (dx(f^{-1}(U) \cap \mathbf{Z}_p^k) / dnU),$$

whenever this limit exists. Clearly r has support contained in \mathbf{Z}_p . We can also consider the Gauss-Weil transform of $1_{(\mathbf{Z}_p)^k}$ by f given by

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p^k} \langle f(x), m \rangle dx.$$

The relationship between these representation masses and the ones introduced in the preceding section is given in the following

LEMMA 2.1. i) Let $n \in \mathbf{Z}_p$, $n \neq 0$, and $t > v_p(4n)$. Then

$$r(n, f, \mathbf{Z}_p) = \lim_{s \rightarrow \infty} p^{(1-k)s} r(n, f, p^s) = p^{(1-k)t} r(n, f, p^t).$$

ii) Let $m \in \mathbf{Z}_p$ and $u \in \mathbf{Z}_p$, $t \geq 1$ be chosen arbitrarily satisfying $m = up^{-t}$. Then

$$\theta(m, f, \mathbf{Q}_p) = p^{-kt} \theta(u, f, p^t).$$

Proof. i) Let $U_t = n + p^t \mathbf{Z}_p$. We have $dn(U_t) = p^{-t}$ and

$$dx(f^{-1}(U_t) \cap \mathbf{Z}_p^k) = \sum_{a \in (\mathbf{Z}/p^t \mathbf{Z})^k} dx(f^{-1}(U_t) \cap (a + p^t \mathbf{Z}_p^k)) = p^{-kt} r(n, f, p^t),$$

since $f^{-1}(U_t) \cap (a + p^t \mathbf{Z}_p^k)$ is equal to $a + p^t \mathbf{Z}_p^k$ or vacuous, according to $f(a) \equiv n \pmod{p^t}$ or not. This proves the first equality in i).

We want now to show that $p^{(1-k)s} r_{p^s}(n) = p^{(1-k)(s-1)} r_{p^{s-1}}(n)$, for all $s > t$. We know that

$$r(n, f, p^s) = p^{-s} \sum_{u=1}^{p^s} \theta(u, f, p^s) \exp(-2\pi i u n p^{-s}).$$

Let us denote by A and B the sum of the terms satisfying $p \mid u$ and $p \nmid u$, respectively. Clearly $A = p^{k-1} r(n, f, p^{s-1})$; hence, we are reduced to proving $B = 0$. Taking into account the explicit computations of Gauss sums (Proposition 1.1), we can express the sum B as

$$B = \begin{cases} C \sum_{u \in (\mathbf{Z}/p^s \mathbf{Z})^*} \left(\frac{u}{p}\right)^a \exp(-2\pi i u n p^{-s}) & \text{if } p > 2 \\ D \sum_{u \in (\mathbf{Z}/2^s \mathbf{Z})^*} \left(\frac{2}{u}\right)^b \exp\left(\frac{2\pi i u}{8}\right)^c \exp(-2\pi i u n 2^{-s}) & \text{if } p = 2, \end{cases}$$

where C, D, a, b, c depend on f and s , but are independent of u . Now, $\exp(-2\pi i n p^{-s})$ is a primitive p^l -th root of 1 with $l > 1$ if $p > 2$, and $l > 3$ if $p = 2$. One can check that, for any function φ defined on $(\mathbf{Z}/p^m \mathbf{Z})^*$, $m \geq 1$ and ξ any primitive p^l -th root of 1, $l > m$, one has

$$\sum_{u \in (\mathbf{Z}/p^l \mathbf{Z})^*} \varphi(u) \xi^u = 0.$$

In particular, B must be zero.

In order to prove ii) we need only to observe that

$$\begin{aligned} \theta(m, f, \mathbf{Q}_p) &= \int_{\mathbf{Z}_p^k} \exp(2\pi i f(x) u p^{-t}) dx \\ &= \sum_{a \in (\mathbf{Z}/p^t \mathbf{Z})^k} \exp(2\pi i f(a) u p^{-t}) \int_{a + p^t \mathbf{Z}_p^k} dx = p^{-kt} \theta(u, f, p^t). \quad \square \end{aligned}$$

Remark. After Siegel [13], it was very well known that for $n \neq 0$ the values $p^{(1-k)t} r(n, f, p^t)$ become constant for $t > 2v_p(4n)$. Lemma 2.1 shows that the minimum value of t with this property can be taken equal to half of the one found by Siegel.

By Lemma 2.1, $r(\cdot, f, \mathbf{Z}_p)$ is locally constant, hence continuous on \mathbf{Q}_p^* , and $r(n, f, \mathbf{Z}_p) = 0$ if and only if n is not represented by f in \mathbf{Z}_p . The fundamental fact is that r is integrable on \mathbf{Z}_p and θ is its Fourier transform. This is well-known [4]. For the sake of completeness we give a short proof of this result using only the background introduced up to now.

PROPOSITION 2.2. $r \in L^1(\mathbf{Z}_p)$ and

$$\theta(m, f, \mathbf{Q}_p) = \int_{\mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn.$$

Proof. We assume $p > 2$. For $p = 2$ the proof works in the same way with minor modifications left to the reader. Let $m = up^{-s}$, $u \in \mathbf{Z}_p$, $s \geq 0$. For all $t > s$, $\mathbf{Z}_p \setminus p^t \mathbf{Z}_p$ is compact, hence $r(n)$, being continuous, is integrable and we have by Lemma 2.1:

$$\begin{aligned} \int_{\mathbf{Z}_p \setminus p^t \mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn &= \sum_{\substack{a \in \mathbf{Z}/p^t \mathbf{Z} \\ a \neq 0}} \int_{a + p^t \mathbf{Z}_p} r(n, f, \mathbf{Z}_p) \langle n, m \rangle dn \\ &= \sum_{\substack{a \in \mathbf{Z}/p^t \mathbf{Z} \\ a \neq 0}} p^{-kt} r(a, f, p^t) \exp(2\pi i a u p^{-s}) = p^{-kt} (\theta(p^{t-s} u, f, p^t) - r(0, f, p^t)) \\ &= \theta(m, f, \mathbf{Q}_p) - p^{-kt} r(0, f, p^t). \end{aligned}$$

Both assertions of the proposition are consequences of Lebesgue's dominated convergence theorem if $p^{-kt} r(0, f, p^t)$ tends to zero as t tends to infinity. This is checked immediately for $k = 1$. For $k > 1$ it can be easily deduced from (1.1) and the explicit computation of Gauss sums in the preceding section. \square

We are ready to prove a crucial fact for the rest of the paper:

THEOREM 2.3. Let f, g be two non-singular integral p -adic quadratic forms in k variables. If $p = 2$, assume that they are of the same type. The following conditions are equivalent:

- i) $f \sim g$ over \mathbf{Z}_p ,
- ii) $r(\cdot, f, \mathbf{Z}_p) = r(\cdot, g, \mathbf{Z}_p)$,
- iii) $\theta(\cdot, f, \mathbf{Q}_p) = \theta(\cdot, g, \mathbf{Q}_p)$.

Proof. If $f \sim g$ over \mathbf{Z}_p , then $f \sim g$ over $\mathbf{Z}/p^t \mathbf{Z}$ and $r(\cdot, f, p^t) = r(\cdot, g, p^t)$ for all $t \geq 1$. By Lemma 2.1 this implies ii). By Proposition 2.2, ii) implies iii). Again by Lemma 2.1, iii) implies that $\theta(\cdot, f, p^t) = \theta(\cdot, g, p^t)$ for all $t \geq 1$, therefore condition i) follows now from Theorem 1.2. \square

Let K be a local field and f a non-singular quadratic form in k variables defined over K . If ϕ is a Schwartz-Bruhat function on K^k , the representation mass function $r_\phi(\cdot, f, K)$ defined as in (0.1) coincides with another classical representation mass function introduced by Weil. This is Weil's procedure (see [4] for the details): for $n \neq 0$, the $(k-1)$ -differential forms

$$\omega_i(x) = (-1)^{i-1} (D_i f)^{-1} dx_1 \wedge \dots \wedge d\hat{x}_i \wedge \dots \wedge dx_k,$$

induce a gauge form ω_n on the affine variety $f^{-1}(n)$. Since we are in a local field, ω_n induces a positive measure $|\omega_n|$ on $f^{-1}(n)$ such that for every continuous function ϕ on K^k with compact support not containing zero we have

$$(2.1) \quad \int_{K^k} \phi(x) dx = \int_K \left(\int_{f^{-1}(n)} \phi |\omega_n| \right) dn.$$

The representation mass of $n \in K^*$ by f with respect to ϕ is then defined as

$$F_\phi(n) = \int_{f^{-1}(n)} \phi |\omega_n|.$$

This function is continuous and after (2.1) it is easy to prove that F_ϕ is integrable and its Fourier transform coincides with the Gauss-Weil transform:

$$\int_{K^k} \phi(x) \langle f(x), m \rangle dx = \int_K F_\phi(n) \langle n, m \rangle dn.$$

Let now $n_o \in K^*$ and let U be any open neighbourhood of n_o . From (2.1) it is also easy to justify that:

$$\int_{f^{-1}(U)} \phi(x) dx = \int_U F_\phi(n) dn.$$

Since F_ϕ is continuous and K is locally compact, we have also:

$$F_\phi(n_o) = \lim_{U \rightarrow \{n_o\}} \left(\int_U F_\phi(n) dn / \int_U dn \right) = r_\phi(n_o),$$

thus $F_\phi = r_\phi$ on K^* .