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The basic philosophy being that flabby sheaves are acyclic for local cohomology, [5] II. 9.3. Thus we can calculate the cohomology sequence (5.1) from the short exact sequence

$$0 \leftarrow \Gamma(U, \Omega^{\bullet \vee \vee}) \xleftarrow{j^*} \Gamma(X, \Omega^{\bullet \vee \vee}) \leftarrow \Gamma_Z(X, \Omega^{\bullet \vee \vee}) \leftarrow 0.$$

According to formula (2.4) we may identify the arrow marked  $j^*$  with the linear dual of the arrow marked  $j_*$ . Simple evaluation according to (2.4) will be written

$$\langle T, l \rangle, \quad T \in \Gamma_c(X, \Omega^{\bullet \vee}), \quad l \in \Gamma(X, \Omega^{\bullet \vee \vee}).$$

This notation is compatible with the symbol introduced in section 1 taking the biduality morphism (2.6) into account. We leave the remaining details with the reader. Q.E.D.

## 6. POINCARÉ DUALITY

Let  $X$  be a  $n$ -dimensional *oriented* smooth manifold. A compactly supported  $(n-p)$ -form  $\alpha$  on  $X$  defines a compact  $p$ -chain  $P\alpha$  given by

$$(6.1) \quad \langle P\alpha, \beta \rangle = \int_X \alpha \wedge \beta, \quad \beta \in \Gamma(X, \Omega^p).$$

(6.2) **THEOREM.** *For a smooth oriented  $n$ -dimensional manifold  $X$ , the transformation  $P$  induces an isomorphism*

$$P: H_c^{n-p}(X, \mathbf{C}) \rightarrow H_p^c(X, \mathbf{C}), \quad p \in \mathbf{N},$$

*from de Rham cohomology with compact support to de Rham homology.*

*Proof.* The following diagram is commutative

$$(6.3) \quad \begin{array}{ccc} \Gamma_c(X, \Omega^{n-p}) & \xrightarrow{P} & D_p^c(X, \mathbf{C}) \\ \downarrow (-1)^n d & & \downarrow (-1)^{p-1} b \\ \Gamma_c(X, \Omega^{n-p+1}) & \xrightarrow{P} & D_{p-1}^c(X, \mathbf{C}) \end{array}$$

as it follows from the relation

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{n-p} \alpha \wedge d\beta, \quad \alpha \in \Gamma_c(X, \Omega^{n-p}), \beta \in \Gamma(X, \Omega^p),$$

using that  $\int d(\alpha \wedge \beta) = 0$ . Upon replacing  $X$  by an arbitrary open subset we obtain a morphism of complexes of sheaves

$$(6.4) \quad P: \Omega^\bullet[n] \rightarrow \Omega^\bullet \vee$$

with the signs of the differentials modified according to the commutative diagram (6.3). The morphism (6.4) is a quasi-isomorphism as it follows by checking the case  $X = \mathbf{R}^n$  by means of the Poincaré lemma with and without compact support. As in the proof of (2.1) we conclude that  $P$  induces a quasi-isomorphism

$$P: \Gamma_c(X, \Omega^\bullet[n]) \rightarrow \Gamma_c(X, \Omega^\bullet \vee).$$

The second complex may be identified with  $D^c(X, \mathbf{C})$  as we have seen in (2.3) and the result follows by passing to homology. Q.E.D.

Let us extend Poincaré duality to the relative groups of an open subset  $U$  of  $X$  with complement  $Z$  in  $X$ . With the notation of (6.1), the operator  $P$  from (6.4) induces a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \Gamma_c(U, \Omega^\bullet[n]) & \rightarrow & \Gamma_c(X, \Omega^\bullet[n]) & \rightarrow & \Gamma_c(Z, \Omega^\bullet[n]) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow P \\ 0 & \rightarrow & D^c(U, \mathbf{C}) & \xrightarrow{j_*} & D^c(X, \mathbf{C}) & \rightarrow & D^c(X, U; \mathbf{C}) \rightarrow 0. \end{array}$$

Again the differentials in the bottom row must be modified as in (6.3). The unmarked vertical arrows are the quasi-isomorphisms of Poincaré duality. The vertical arrow marked  $P$  is induced by the algebra of the diagram. Again, from algebra we deduce a *quasi-isomorphism*

$$(6.5) \quad P: \Gamma_c(Z, \Omega^\bullet[n]) \rightarrow D^c(X, U; \mathbf{C}), \quad Z = X - U.$$

Passing to homology we obtain the Poincaré duality isomorphism

$$(6.6) \quad P: H_c^{n-p}(Z, \mathbf{C}) \xrightarrow{\sim} H_p^c(X, U; \mathbf{C}).$$

The  $p$ 'th sheaf cohomology group with compact support  $H_c^p(Z, \mathbf{C})$  has the following de Rham representation

$$(6.7) \quad \left\{ \omega \in \Gamma_c(X, \Omega^p) \mid \text{Supp}(d\omega) \subseteq U \right\} \Big/ \begin{array}{l} \left\{ d\nu \mid \nu \in \Gamma_c(X, \Omega^{p-1}) \right\} \\ + \left\{ \omega \in \Gamma_c(X, \Omega^p) \mid \text{Supp}(\omega) \subseteq U \right\} \end{array}$$

as it follows from the exact sequence making up the top row of the diagram above.