

# 4. Symplectic HR-matrices

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **29.06.2024**

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## 4. SYMPLECTIC HR-MATRICES

4.1. Symplectic matrices  $A$  leave invariant the bilinear form with coefficient matrix  $J = \begin{pmatrix} & E_n \\ -E_n & \end{pmatrix}$ ; i.e.,  $A^T J A = J$ . With respect to the HR-matrix relations (1) they behave exactly like orthogonal or unitary matrices:

PROPOSITION 4.1. Let  $A_1, A_2, \dots, A_s$  be  $2n \times 2n$ -matrices, and  $A_0 = E_{2n}$ . Then  $\sum_0^s x_j A_j$  is symplectic up to the factor  $\sum_0^1 x_j^2$  for all  $x_0, x_1, \dots, x_s$  if and only if  $A_1, A_2, \dots, A_s$  is a set of symplectic HR-matrices.

*Proof.* 
$$\left( \sum_0^s x_j A_j^T \right) J \left( \sum_0^s x_j A_j \right) = \sum_0^s x_j^2 A_j^T J A_j$$

$$+ \sum_1^s x_0 x_j (A_j^T J + J A_j) + \sum_{j,k=1}^s x_j x_k (A_j^T J A_k + A_k^T J A_j), \quad j \neq k.$$

Assume  $A_j^T J A_j = J, j = 0, \dots, s$ ; and

$$A_j^2 = -E, A_j A_k + A_k A_j = 0, j, k = 1, \dots, s, j \neq k.$$

Then  $-A_j^T J = J A_j$ , and  $A_j^T J A_k + A_k^T J A_j = -J(A_j A_k + A_k A_j) = 0$ . Thus the whole expression reduces to  $\left( \sum_0^s x_j^2 \right) J$ . The argument is plainly reversible.

4.2. In the following, "symplectic" will mean unitary symplectic; i.e., we consider matrices from the compact group  $Sp(n) \subset U(2n)$ . A set of symplectic HR-matrices  $A_1, A_2, \dots, A_s$  is thus an  $\varepsilon$ -representation of  $G_s$  in  $Sp(n)$ ; we continue to call its degree  $2n$ . The notations  $v_s^{Sp}, d_s^{Sp}, D_s^{Sp}, E_s^{Sp}$  have the same meaning as before for  $U$  and for  $O$ .

All elements of  $G_s$  have square 1 or  $\varepsilon$ ; a matrix  $\in U(2n)$  of square  $\pm E$  is symplectic if and only if it is of the form  $\begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}$  with  $B^t = -B$ ,  $\bar{A}^T = A$  in the case of square  $E$ , and  $B^t = B$ ,  $\bar{A}^t = -A$  in the case of square  $-E$ . Symplectic representations of  $G_s$  are sums of irreducible unitary representations; if an irreducible unitary  $\varepsilon$ -representation is not (equivalent to a) symplectic, we have to add its conjugate-complex in order to obtain an irreducible symplectic  $\varepsilon$ -representation. Due to the description (2) of the  $G_s$  the following observations yield the complete list of degrees etc.

4.3. (a) The tensor product of a unitary representation  $V$  of even degree and an orthogonal representation (of any degree) is symplectic if and only if  $V$  is.

(b) Since  $Sp(1) = SU(2)$ , the irreducible unitary  $\varepsilon$ -representations (of degree 2) of  $G_2 = Q$  are symplectic.

(c) The irreducible  $\varepsilon$ -representations of  $D$  (= dihedral group of order 8) are not symplectic, but orthogonal; the same holds for  $D^j$  and  $D^jK$ ,  $K =$  Klein 4-group.

(d) The tensor product of any representation with the irreducible  $\varepsilon$ -representation (of degree 1) of  $G_1 = C$  is not symplectic.

The periodicity modulo 8,  $G_{s+8} = G_8G_s = D^4G_s$ , with  $d_8^O = d_8^U = 16$ , yields  $d_{s+8}^{Sp} = 16d_s^{Sp}$  and  $v_{s+8}^{Sp} = v_s^{Sp}$ . For  $s \equiv 2, 3, 4$  modulo 8 the irreducible unitary  $\varepsilon$ -representations of  $G_s$  are symplectic,  $d_s^{Sp} = d_s^U$  and  $v_s^{Sp} = v_s^U$ ; for the other  $s$  they are not, thus  $d_s^{Sp} = 2d_s^U$ . For  $s \equiv 1, 5$  modulo 8 the conjugate-complex representations are inequivalent, thus  $v_s^{Sp} = 1$ ; for  $s \equiv 0, 6, 7$  we combine two equivalent representations, thus  $v_s^{Sp} = v_s^U$ , i.e.,  $v_s^{Sp} = 1$  for  $s \equiv 0, 6$  and  $v_s^{Sp} = 2$  for  $s \equiv 7$ . The restriction arguments from  $G_{s+1}$  to  $G_s$  are as before and yield the  $E_s^{Sp}$ , which are periodic modulo 8.

We summarize the results in the following table

(6) $s$	0	1	2	3	4	5	6	7	8	9
$v_s^{Sp}$	1	1	1	2	1	1	1	2	1	1
$d_s^{Sp}$	2	2	2	2	4	8	16	16	32	32
$D_s^{Sp}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z} \oplus \mathbf{Z}$	$\mathbf{Z}$	$\mathbf{Z}$
$E_s^{Sp}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0

4.4. Comparing with (3) one notes that  $D_s^O \cong D_{s+4}^{Sp}$  and  $E_s^O \cong E_{s+4}^{Sp}$ . The isomorphisms can be made explicit in terms of the  $\cup$ -product introduced in 2.2, as follows.

Let  $\rho_3 \in D_3^U = D_3^{Sp}$  be one of the generators,  $\rho_3 = \bar{\rho}_3$ , and  $\sigma_t \in D_t^O$  one of the generators. The product  $\rho_3 \cup \sigma_t \in D_{t+4}^U$  has degree  $2.2.d_t^O$ ; this is precisely the degree of a generator of  $D_{t+4}^{Sp}$ . We check that  $\rho_3 \cup \sigma_t$  is indeed in  $D_{t+4}^{Sp}$  and thus a generator: this is clear for  $t \equiv 0, 6, 7$ ,  $t + 4 \equiv 2, 3, 4$  modulo 8 where  $D_{t+4}^{Sp} = D_{t+4}^U$ ; for  $t \equiv 1, 2, 3, 4, 5$  we know that  $\sigma_t = \rho_t + \bar{\rho}_t$ , whence  $\rho_3 \cup \sigma_t = \rho_3 \cup \rho_t + \bar{\rho}_3 \cup \bar{\rho}_t$ , i.e., it is one of the generators of  $D_{t+4}^{Sp}$ .

THEOREM 4.1. *The product of the generator  $\rho_3 \in E_3^U = E_3^{Sp}$  with  $E_s^O$  is an isomorphism  $E_s^O \cong E_{s+4}^{Sp}$  for all  $s \geq 0$ .*

4.5. We now consider the homomorphism  $\theta: E_s^{Sp} \rightarrow \pi_s(Sp)$ , analogous to  $\phi$  and  $\psi$  before.

Let  $A_1, A_2, \dots, A_s$  be a set of  $s$  symplectic  $2n \times 2n$  HR-matrices, and  $A_0 = E$ . Then

$$f_s(x_0, x_1, \dots, x_s) = \sum_0^s x_j A_j$$

$x = (x_0, x_1, \dots, x_s) \in \mathbf{R}^{s+1}$ ,  $\sum_0^s x_j^2 = 1$ , is symplectic. We consider  $f_s$  as a map  $S^s \rightarrow Sp$  via  $Sp(n)$ ; as in the cases  $U$  and  $O$  this yields a homomorphism  $\theta: E_s^{Sp} \rightarrow \pi_s(Sp)$ ,  $s \geq 0$ . The  $\pi_s(Sp)$  are known to be 0 or cyclic. Theorem A' can now be reformulated as follows.

THEOREM B'.  *$\theta$  is an isomorphism  $E_s^{Sp} \rightarrow \pi_s(Sp)$ ,  $s \geq 0$ .*

For  $s = 3$  this is clear: since  $E_3^{Sp} = E_3^U$  and  $\pi_3(Sp) \cong \pi_3(Sp(1)) = \pi_3(SU(2)) \cong \pi_3(U)$ ,  $c = \theta(\rho_3)$  is a generator of  $\pi_3(Sp) = \mathbf{Z}$ .

To complete the proof of Theorem B' we use, as for Theorem B, the  $\cup$ -product and results of  $K$ -theory relating  $K_{\mathbf{R}}$  with  $K_{\mathbf{H}}$ , the quaternionic or symplectic  $K$ -theory. The product  $c \cup b$ ,  $b \in \pi_s(O)$ , can be expressed in terms of linear maps  $S^3 \rightarrow Sp(1) = SU(2)$ ,  $S^s \rightarrow O(m)$ ,  $S^{s+4} \rightarrow U(4m)$ . As seen in 4.3, it lies in fact in  $Sp(2m) \subset U(4m)$  and can thus be regarded as an element of  $\pi_{s+4}(Sp)$ . The map  $c \cup - : \pi_s(O) \rightarrow \pi_{s+4}(Sp)$  corresponds, under  $\pi_s(O) \cong \tilde{K}_{\mathbf{R}}(S^{s+1})$  and  $\pi_t(Sp) \cong \tilde{K}_{\mathbf{H}}(S^{t+1})$ , to the isomorphism  $\tilde{K}_{\mathbf{R}}(S^{s+1}) \rightarrow \tilde{K}_{\mathbf{H}}(S^{s+5})$  given by the external tensor product of bundles with the generating bundles of  $\tilde{K}_{\mathbf{H}}(S^4) = \mathbf{Z}$  (see [K], p. 154). Hence  $c \cup -$  is an isomorphism  $\pi_s(O) \cong \pi_{s+4}(Sp)$ .

Moreover, since everything is described by linear maps the diagram

$$\begin{array}{ccc} E_s^O & \xrightarrow{\psi} & \pi_s(O) \\ \rho_3 \cup - \downarrow & & \downarrow c \cup - \\ E_{s+4}^{Sp} & \xrightarrow{\theta} & \pi_{s+4}(Sp) \end{array}$$

is commutative. The upper and the two vertical maps being isomorphisms, so is  $\theta$ .