

# 5. LOCALLY LINEAR REPRESENTATION

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **35 (1989)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **05.08.2024**

## **Nutzungsbedingungen**

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

## **Haftungsausschluss**

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

$\text{Fix}(A_4)$  is a sphere. It cannot be  $S^2$  since the representation of  $A_4$  in  $SO(3)$  is irreducible, so it is  $S^1$ . The only closed 1-dimensional submanifold of  $S^1$  is  $S^1$  itself, so  $\text{Fix}(G) = S^1$ .

b. As in subcase a., a linear change in coordinates allows us to assume that  $h$  is actually  $\tilde{i}$ , and as before if  $G_2 \in G$  the proposition is proved applying 4.1.

If it is not the case, let  $\alpha$  correspond to the cycle  $(12345) \in A_5$ ,  $\beta$  to  $(123)$  and  $\gamma$  to  $(345)$ . We observe that  $\beta$  and  $\gamma$  generate  $A_5$  and so:

1.  $\text{Fix}(A_5) = \text{Fix}(\beta) \cap \text{Fix}(\gamma)$ ,
2.  $\text{Fix}(A_5) \subset \text{Fix}(\alpha)$ .

We claim that  $\text{Fix}(\alpha)$  is  $S^0$ . According to Smith's theorem it is enough to prove that the representation of  $\alpha$  around  $x_0$  has an isolated fixed point, i.e. is the sum of two irreducible complex ones.

If not by Lemma 3.3  $(\bar{i}(\alpha); i(\alpha))$  would be conjugate in  $SO(3) \times SO(3)$  to an element on the diagonal. From the explicit description of  $i$  and  $\bar{i}$  (see the end of section 7.1 of [22]), it follows that they send all the five cycles to non conjugate elements in  $SO(3)$ , so this is impossible, and  $\text{Fix}(\alpha) = S^0$ .

As for  $\beta$  and  $\gamma$ , their images under  $(\bar{i}, i)$  are conjugate to elements on the diagonal, by 3.3 and 3.4 their fixed point sets have two-dimensional components, and so by Smith's theorem they are copies of  $S^2$ .

So  $\text{Fix}(G)$  is the intersection of a couple of  $S^2$ s and is contained in  $\text{Fix}(\alpha)$  which is  $S^0$ . If this set is empty or equal to  $S^0$ , the proposition follows. If it were a single point, it would be a transverse intersection, by local linearity, but it is not possible since a homology  $S^4$  does not contain any two cycles with intersection number odd. This ends the proof.

## 5. LOCALLY LINEAR REPRESENTATION

Let's now consider the case of  $G$  acting on a homology  $S^4$  with two fixed points,  $P_0$  and  $P_1$ .

**THEOREM 5.1.** *The unoriented representations of  $G$  around  $P_0$  and  $P_1$  are linearly equivalent.* <sup>1)</sup>

*Proof.* It will suffice to show that the characters associated to the representations around the  $P_i$ s agree on every cyclic subgroup  $C_k$  of  $G$ .

<sup>1)</sup> See the note in the introduction.

Observe that by Lemma 3.4 and Smith's theorem the fixed point set of an element of  $G$  different from the identity is either  $S^0$  or  $S^2$ .

Let  $g$  generate  $C_k$ , we distinguish three cases:

1.  $\text{Fix}(g^r) = \{P_1; P_2\}$  for every  $r \equiv 0(\text{mod } k)$ ,
2.  $\text{Fix}(g) = S^2$ ,
3.  $\text{Fix}(g) = \{P_1; P_2\}$  but  $\text{Fix}(g^n) = S^2$  for some  $g^n \neq \text{id}$ .

Case 1. The hypothesis means that the action is semifree and the claim follows from the work of Atiyah and Bott, see [1] and [14].

Case 2. The action of  $C_k$  on the normal bundle of the fixed  $S^2$  defines an element  $N$  of  $K_{C_k}(S^2)$ . Since  $C_k$  acts trivially on  $S^2$  the two inclusions  $P_i \rightarrow S^2$  are obviously  $C_k$  homotopic so that the diagram:

$$\begin{array}{ccccc}
 & & K_{C_k}(P_2) & & \\
 & \nearrow & & \searrow & \\
 [N] \in K_{C_k}(S^2) & & & & R(C_k) \\
 & \searrow & & \nearrow & \\
 & & K_{C_k}(P_1) & & 
 \end{array}$$

commutes. This means that the representation of  $C_k$  in the normal component to  $S^2$  are conjugate, the tangential representations are of course both the identity, so the statement is proved.

Case 3. We can assume, by [8], that the action on  $S^2 = \text{Fix}(g^n)$  is linear.  $S^2$  has zero intersection number in  $\Sigma$  so its normal bundle  $N$  can be identified to  $S^2 \times R^2$ , and we fix a trivialization. Denote a point of  $S^2 - \{P_1; P_2\}$  by  $(x, t)$  with  $x \in S^1$  and  $t \in (0, 1)$ . Let  $C_0$  be the space  $\{\phi: S^1 \rightarrow SO(2) \mid \text{deg } \phi = 0\}$ , it is an abelian group by pointwise multiplication and a  $C_k$  module with structure given by:

$$(h\phi)(x) = \phi(hx), \quad h \in C_k \quad \text{and} \quad x \in S^1 \subset S^2$$

acted on by the obvious induced action.

By [5], chapter VI, prop. 11.1, the action is given by a  $\theta_t$  such that

1.  $\theta_t \in Z^1(C_k; C_0)$  and depends continuously on  $t \in [0, 1]$ .
2.  $\theta_i(h)(x)$  is constant on  $x \in S^1$  and equal to the representation of  $h$  at  $P_i$  for  $i = 0; 1$ .

A change in the trivialization adds to each  $\theta_t$  a coboundary so there is a well defined continuous family  $\theta_t: [0, 1] \rightarrow H^1(C_k; C_0)$ .

A straightforward calculation shows that  $H^1(C_k; C_0) = H^2(C_k; Z) = C_k$ . Since  $\theta_t$  is continuous it has to be constant, so  $\theta_0 = \theta_1$  and by 2. the

two normal representations are equal. In the topological case, by the results of Cappel and Shaneson topological equivalence of matrices in dimension 4 implies linear equivalence, so the statement of Theorem 5.1 makes sense also for a group of homeomorphism.

The proof given can be adapted to this more general case provided that the followings are true:

1. the topological Atiyah-Singer signature formula holds,
2. a locally flat  $S^2$  in  $\Sigma$  has a normal bundle,
3. the argument in case 3 works with  $\text{Homeo}(S^1)$  instead of  $SO(2)$ .

Assertion 1 is proved, in the case of the semi-free action, in [21], page 188; assertion 2 follows from the work of Freedman, see [10]; assertion 3 is proved using the retraction  $\text{Homeo}(S^1)$  into  $SO(2)$  given by the Poincaré number, see [7].

APPENDIX

LEMMA. *The extensions:*

$$\begin{array}{ccccccccc}
 0 & \rightarrow & C_2 & \rightarrow & \tilde{A}_5 & \rightarrow & A_5 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & C_2 & \rightarrow & A_5 \times A_5 & \rightarrow & A_5 \times A_5 & \rightarrow & 0 \\
 & & & & \downarrow & & \downarrow^{(h, h')} & & \\
 0 & \rightarrow & C_2 & \rightarrow & SO(4) & \rightarrow & SO(3) \times SO(3) & \rightarrow & 0
 \end{array}$$

are not split,  $h$  and  $h'$  can be any nontrivial representations of  $A_5$  and  $f$  is either  $(Id \times \{I\})$  or  $(\{I\} \times Id)$ .

*Proof.* Standard theory of group extensions and cohomology (see [4]) allows us to reduce to the:

PROPOSITION. *Any non trivial homomorphism  $A_5 \xrightarrow{i} SO(3)$  induces an isomorphism  $Z/2 = H^2(BSO(3); Z/2) \xrightarrow{i} H^2(BA_5; Z/2) = Z/2$ .*

*Proof of the Proposition.* If the corresponding extension is split, then  $Z/2 \times A_5 \subset S^3$ , but  $A_5 = 60$  so there exists a  $Z/2 \subset A_5$  so  $Z/2 \times Z/2$  would act freely on  $S^3$ , which cannot happen.