5. R-MATRICES AND INTERTWINING OPERATORS

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5. R-MATRICES AND INTERTWINING OPERATORS

In this section we shall prove that, after a trivial twisting, the intertwining operators between certain representations of Yangians provide rational solutions of the quantum Yang-Baxter equation. Recall that, if V is any representation of $Y = Y(\mathfrak{sl}_2)$, then, for any $a \in \mathbf{C}$, we denote by $V(a)$ its pull-back by the automorphism τ_a of Y defined in Proposition 2.5.

PROPOSITION 5.1. *Let V, W be irreducible finite-dimensional representations of Y with highest weight vectors Ω_V, Ω_W and let $a, b \in \mathbf{C}$. Then:*

(a) the tensor products $V(a) \otimes W(b)$ and $W(b) \otimes V(a)$ are irreducible and isomorphic except for a finite set of values $S(V, W)$ of $a - b$;

(b) the unique intertwining operator

$$I(V, a; W, b)\colon W(b) \otimes V(a) \to V(a) \otimes W(b)$$

which maps $\Omega_W \otimes \Omega_V$ to $\Omega_V \otimes \Omega_W$ is a rational function of $a - b$ with values in $\mathrm{Hom}(W \otimes V, V \otimes W)$.

Proof. Part (a) follows immediately from Proposition 4.2 and Corollary 4.7. For part (b), we need the following lemma.

LEMMA 5.2. *Let V, W be representations of Y and let $a \in \mathbf{C}$.*

(a) If V is irreducible, so is $V(a)$.

(b) If $I\colon V \to W$ is an isomorphism of representations of Y, so is $I\colon V(a) \to W(a)$.

Proof of lemma. Part (a) follows from the definition of $V(a)$. For part (b), we must show that I commutes with the action of x and $J(x)$ on $V(a)$ and $W(a)$, for all $x \in \mathfrak{sl}_2$. But this is clear, since the action of x is the same as that on V and W, and that of $J(x)$ is the same as that of $J(x) + ax$ on V and W.

Returning to the proof of Proposition 5.1, it follows from the lemma that $I(V, a; W, b)$ is a function of $a - b$, so it suffices to consider the case $b = 0$. For any $a \in \mathbf{C}$ which does not belong to the finite set $S(V, W)$, there is a unique isomorphism

$$I(V, a; W, 0) \equiv I(a)\colon W \otimes V(a) \to V(a) \otimes W$$

of representations of Y such that

$$I(a)\,(\Omega_W \otimes \Omega_V) = \Omega_V \otimes \Omega_W\,. \tag{5.3}$$

Choose bases of $V \otimes W$ and $W \otimes V$ and let $\{I_\lambda\}$ be a basis of $\mathfrak{sl}_2$; write $I(a)$ also for the matrix of $I(a)$ with respect to these bases. Let A_λ, B_λ be the matrices of I_λ and $J(I_\lambda)$ acting on $W \otimes V(a)$; and let A'_λ and B'_λ refer similarly to $V(a) \otimes W$. Then, $I(a)$ commutes with the action of Y if and only if $I(a)$ satisfies the following system of homogeneous linear equations:

$$A_\lambda I(a) = I(a) A'_\lambda, \quad B_\lambda I(a) = I(a) B'_\lambda, \quad \text{for all} \quad \lambda .$$

We know that, if $a \notin S(V, W)$, these equations have a unique solution satisfying equation (5.3). By elementary linear algebra, the solution is a rational function of the entries of the matrices $A_\lambda, A'_\lambda, B_\lambda, B'_\lambda$. Since A_λ, A'_λ are independent of a and B_λ, B'_λ are linear in a, the result follows.

Definition 5.4. Let V be a finite-dimensional irreducible representation of Y. Then, the *R-matrix associated to* V is the function $R(a-b)$ with values in $\mathrm{End}(V \otimes V)$ given by

$$R(a-b) = I(V, a; V, b)\sigma ,$$

where $\sigma \in \mathrm{End}(V \otimes V)$ is the switch of the two factors.

THEOREM 5.5. *Let V be a finite-dimensional irreducible representation of Y. Then the R-matrix associated to V is a rational solution of the quantum Yang-Baxter equation:*

$$R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) = R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) . \tag{5.6}$$

Proof. We note first some simple commutation relations between the intertwining operator $I(a-b) \equiv I(V, a; V, b)$ and the switch map σ. For example, we have

$$\sigma^{12} I^{13}(a-c)\sigma^{12} = I^{23}(a-c) .$$

by an easy computation. Similarly,

$$\sigma^{12}\sigma^{13} I^{23}(b-c)\sigma^{13}\sigma^{12} = I^{12}(b-c) .$$

Hence,

$$\begin{aligned} R^{12}(a-b)R^{13}(a-c)R^{23}(b-c) &= I^{12}(a-b)\sigma^{12} I^{13}(a-c)\sigma^{13} I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b) I^{23}(a-c)\sigma^{12}\sigma^{13} I^{23}(b-c)\sigma^{23} \\ &= I^{12}(a-b) I^{23}(a-c) I^{12}(b-c)\sigma^{12}\sigma^{13}\sigma^{23} . \end{aligned}$$

Similarly,

$$R^{23}(b-c)R^{13}(a-c)R^{12}(a-b) = I^{23}(b-c) I^{12}(a-c) I^{23}(a-b)\sigma^{23}\sigma^{13}\sigma^{12} .$$

Hence, in view of the relation

$$\sigma^{12}\sigma^{13}\sigma^{23} = \sigma^{23}\sigma^{13}\sigma^{12}$$

in the symmetric group on three letters, the equation to be proved is

$$I^{12}(a-b)I^{23}(a-c)I^{12}(b-c) = I^{23}(b-c)I^{12}(a-c)I^{23}(a-b)\ . \tag{5.7}$$

Note that both sides of equation (5.7) define intertwining operators

$$V(c) \otimes V(b) \otimes V(a) \to V(a) \otimes V(b) \otimes V(c)$$

which fix the tensor product of the highest weight vectors in V. Hence, regarded as functions on $\mathbf{C}^3$ with values in $\mathrm{End}(V \otimes V \otimes V)$, they agree on the complement of the set S of $(a, b, c) \in \mathbf{C}^3$ where $V(c) \otimes V(b) \otimes V(a)$ or $V(a) \otimes V(b) \otimes V(c)$ is reducible. It follows from part (a) of Proposition 5.1 that S intersects each complex line parallel to one of the axes in $\mathbf{C}^3$ in at most finitely many points. It is easy to see that the complement of such a set is Zariski dense in $\mathbf{C}^3$. Since the two sides of equation (5.7) are rational functions which agree on a Zariski dense set, they are equal.

Remark. We have used the following simple fact about intertwining operators. Let U, V and W be representations of a Yangian $Y(\mathfrak{sl}_2)$ and let $I: U \otimes V \to V \otimes U$ be an intertwining operator. Then

$$I^{12}: U \otimes V \otimes W \to V \otimes U \otimes W$$

and

$$I^{23}: W \otimes U \otimes V \to W \otimes V \otimes U$$

are intertwining operators. While this is obvious enough, it should be noted that

$$I^{13}: U \otimes W \otimes V \to V \otimes W \otimes U$$

is *not* an intertwining operator in general.

We conclude this general discussion by showing that, up to a sign change in the parameter, the R-matrix $R(u)$ we have associated to a representation of Y is the same as that constructed using the "universal R-matrix" (see Theorem 3 of [4]). Set

$$\tilde{R}(u) = R(-u)\ .$$

Then, by Theorem 4 of [4], it suffices to prove that

$$P_\lambda^+(a, b)\tilde{R}(b-a) = \tilde{R}(b-a)P_\lambda^-(a, b) \tag{5.8}$$

where

$$P_\lambda^{\pm}(a, b) = (\rho \otimes \rho)\,((J(I_\lambda) + aI_\lambda) \otimes 1 + 1 \otimes (J(I_\lambda) + bI_\lambda) + \frac{1}{2}[I_\lambda \otimes 1, \Omega]),$$

$\rho: Y \to \mathrm{End}(V)$ is the action of Y on V and $\{I_\lambda\}$ is an orthonormal basis of $\mathfrak{sl}_2$. In terms of intertwining operators, equation (5.8) asserts that

$$P_\lambda^+(a, b) I(a-b) = I(a-b)\sigma P_\lambda^-(a, b)\sigma .$$

But it is easy to see that

$$\sigma P_\lambda^-(a, b)\sigma = P_\lambda^+(b, a) .$$

Hence, we must prove that

$$P_\lambda^+(a, b) I(a-b) = I(a-b) P_\lambda^+(b, a) .$$

But this is simply the statement that

$$I(a-b): V(b) \otimes V(a) \to V(a) \otimes V(b)$$

commutes with the action of $J(I_\lambda)$.

We shall now apply these results to compute the R-matrices associated to every finite-dimensional irreducible representation of Y. By Theorem 4.11, every such representation is of the form

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k).$$

The intertwining operator

$$I(a-b): V(b) \otimes V(a) \to V(a) \otimes V(b)$$

can be computed as the product of k^2 intertwining operators of the form $I(V_m, a; V_n, b)$, each of which effects an interchange of nearest neighbours. Since such an operator commutes, in particular, with the action of $\mathfrak{sl}_2$, it can be written in the form

$$I(V_m, a; V_n, b) = \sum_{j=0}^{\min\{m,n\}} c_j P_{m+n-2j}, \tag{5.9}$$

where

$$P_{m+n-2j}: V_n \otimes V_m \to V_m \otimes V_n$$

is the projection onto the irreducible component of

$$V_m \otimes V_n \cong \otimes_{j=0}^{\min\{m,n\}} V_{m+n-2j}$$

of type V_{m+n-2j}. We have $c_0 = 1$ since $I(V_m, a; V_n, b)$ preserves the tensor products of the highest weight vectors.

To compute $I(V_m, a; V_n, b)$, let $\Omega_j, j = 0, 1, \ldots, \min\{m, n\}$, be a highest weight vector in $V_n \otimes V_m$ of weight $m + n - 2j$; then, the vector Ω_j' obtained by switching the order of the factors in Ω_j is a highest weight vector in $V_m \otimes V_n$ of the same weight, and we have

$$I(V_m, a; V_n, b)\,(\Omega_j) = \Omega_j' .$$

Further, it is easy to see that, for $j > 0, (x^+ \otimes 1).\Omega_j$ is an $\mathfrak{sl}_2$-highest weight vector of weight $m + n - 2j + 2$; it is non-zero, since otherwise Ω_j would be annihilated by $x^+ \otimes 1$ and by $1 \otimes x^+$, contracting the assumption $j > 0$. Hence, we may assume that

$$(x^+ \otimes 1).\Omega_j = \Omega_{j-1}$$

for $j > 0$. Switching the order of the factors, we have

$$(x^+ \otimes 1).\Omega_j' = -\Omega_{j-1}' .$$

By Proposition 4.2 (and its proof), Ω_j is a Y-highest weight vector in $V_n(b) \otimes V_m(a)$ if

$$b - a = \frac{1}{2}(m+n) - j + 1 .$$

It follows from the formula for the co-multiplication in Definition 1.1 that, in the representation $V_n(b) \otimes V_m(a)$,

$$J(x^+).\Omega_j = \left(b - a - \frac{1}{2}(m + n) + j - 1\right)(x^+ \otimes 1).\Omega_j ,$$

and that in the representation $V_m(a) \otimes V_n(b)$,

$$J(x^+).\Omega_j' = \left(a - b - \frac{1}{2}(m + n) + j - 1\right)(x^+ \otimes 1).\Omega_j' .$$

The equation

$$I(V_m, a; V_n, b)\,(J(x^+).\Omega_j) = J(x^+).(I(V_m, a; V_n, b)\Omega_j)$$

now gives

$$\frac{c_j}{c_{j-1}} = \frac{a - b + \frac{1}{2}(m+n) - j + 1}{a - b - \frac{1}{2}(m+n) - j + 1} .$$

It follows that

$$(5.10)\qquad I(V_m, a; V_n, b) = \sum_{j-0}^{\min\{m,n\}} \prod_{i=0}^{j=1} \frac{a-b+\frac{1}{2}(m+n)-i}{a-b-\frac{1}{2}(m+n)+i} P_j .$$

We summarize our results in the following theorem.

THEOREM 5.11. *The R-matrix associated to the representation*

$$V = V_{m_1}(a_1) \otimes \cdots \otimes V_{m_k}(a_k)$$

of Y *is given by*

$$R(a-b) = \Big(\prod_{i,j=1}^{k} I(V_{m_i}, a+a_i; V_{m_j}, b+a_j)\Big)\sigma ,$$

where the intertwining operators are given by equation (5.10) and σ *is the switch map. The order of the factors in the product is such that the* (i,j)*-term appears to the left of the* (i',j')*-term iff*

$$i > i' \quad or \quad i = i' \quad and \quad j < j' .$$

6. CONCLUDING REMARKS

Since we have discussed only the Yangian associated to $\mathfrak{sl}_2$ in this paper, it may be worth-while to indicate the extent to which the results above can be generalized to the Yangian $Y(\mathfrak{a})$ associated to an arbitrary finite-dimensional complex simple Lie algebra $\mathfrak{a}$.

The definition of $Y(\mathfrak{a})$ is precisely as in (1.1), except of course that $\{I_\lambda\}$ should be an orthonormal basis of $\mathfrak{a}$ with respect to some invariant inner product. The formulae

$$\tau_a(x) = x , \qquad \tau_a(J(x)) = J(x) + ax ,$$

for $x \in \mathfrak{a}$, again define a one-parameter group of Hopf algebra automorphisms of $Y(\mathfrak{a})$, and the relation, discussed in section 5, between solutions of the quantum Yang-Baxter equation and intertwining operators between tensor products of representations of $Y(\mathfrak{a})$, which follows from the existence of the τ_a, is also valid in the general case.