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Artikel: THE DISTANCE BETWEEN IDEALS IN THE ORDERS OF A REAL QUADRATIC FIELD

Autor: Kaplan, Pierre / Williams, Kenneth S.

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Then we have $GCD(a, b, c) = 1$, where $\frac{D - b^2}{4a} = c$, and so $GCD(a, bf, cf^2) = 1$, showing that $I = \left[a, f\left(\frac{b + \sqrt{D}}{2}\right)\right]$ is a primitive ideal of $O_{D'}$. Hence C is the image of the class of I under θ.

COROLLARY 4. *If the class C of $O_{D'}$ contains the primitive ideal $I = \left[a, \frac{b + \sqrt{D'}}{2}\right]$, where $f^2 \mid a$, then $f \mid b$ and the class $\theta(C)$ contains the primitive ideal $J = \left[\frac{a}{f^2}, \frac{\frac{b}{f} + \sqrt{D}}{2}\right]$ of O_D.*

Proof. As $D' = Df^2 = b^2 + 4ac$, and $f^2 \mid a$, we see that $f \mid b$, and so $GCD(f, c) = 1$. By Corollary 2 we have $I = \left(\frac{\sqrt{D'} - b}{2a}\right)\left[c, \frac{-b + \sqrt{D'}}{2}\right]$ and so, by Theorem 1, we see that $\left[c, \frac{-\frac{b}{f} + \sqrt{D}}{2}\right] \in \theta(C)$. Finally, by Corollary 2, $J = \left[\frac{a}{f^2}, \frac{b/f + \sqrt{D}}{2}\right] = \frac{\left(\sqrt{D} + \frac{b}{f}\right)}{2c}\left[c, \frac{\frac{-b}{f} + \sqrt{D}}{2}\right]$, showing that $J \in \theta(C)$.

4. REDUCED IDEALS

From now on in this paper we suppose that $D_0 > 0$ so that we are only considering ideals in orders of a real quadratic field. An ideal I of O_D can be written in the form $I = ad[1, \phi]$, where $\phi = \frac{b + \sqrt{D}}{2a}$. By Proposition 1 (ii), if $I = a'd'[1, \phi']$ is another representation of I, then $a' = \pm a$ and $\phi' \equiv \frac{a}{a'} \phi \pmod 1$. A real number of the form $\frac{b + \sqrt{D}}{2a}$, where $c = \frac{D - b^2}{4a}$ is an integer and $GCD(a, b, c) = 1$ is called a quadratic irrationality of discriminant D.

Definition 9. (Reduced number). The quadratic irrationality $\phi = \dfrac{b+\sqrt{D}}{2a}$ of discriminant D is said to be *reduced* if

$$\phi > 1\,, \quad -1 < \bar{\phi} < 0\,. \tag{4.1}$$

It is easy to check that (4.1) is equivalent to each of the inequalities in (4.2)

$$\text{(i)} \quad 0 < \sqrt{D} - b < 2a < \sqrt{D} + b\,, \tag{4.2}$$

$$\text{(ii)} \quad 0 < \sqrt{D} - b < 2c < \sqrt{D} + b\,.$$

Moreover (4.2) implies

$$0 < a < \sqrt{D}\,, \quad 0 < b < \sqrt{D}\,, \quad 0 < c < \sqrt{D}\,. \tag{4.3}$$

Definition 10. (Reduced ideal). The ideal $I = ad[1, \phi]$ of O_D, where $\phi = \dfrac{b+\sqrt{D}}{2a}$, is said to be *reduced* if, and only if, ϕ can be chosen to be reduced.

From (4.3) we see that the number of reduced, primitive ideals of O_D is finite.

PROPOSITION 4. ([12]: Definition and Theorem 3.5). *The ideal*

$$I = d\left[a, \frac{b+\sqrt{D}}{2}\right]$$

of O_D, *where* $a > 0$ *and* $d > 0$, *is reduced if, and only if,* I *does not contain a nonzero element* α *satisfying* $|\alpha| < da$, $|\bar{\alpha}| < da$.

Proof. It suffices to prove that I is reduced if, and only if, the Z-module $[1, \phi]$ does not contain a nonzero element $\lambda = x + y\phi$ such that

$$|\lambda| < 1\,, \quad |\bar{\lambda}| < 1\,. \tag{4.4}$$

If I is reduced we can suppose that $\phi > 1$, $-1 < \bar{\phi} < 0$. Let x and y be integers such that $0 < \lambda = x + y\phi < 1$.

Clearly we have $y \neq 0$. If $y \geqslant 1$, then we have $y\phi > 1$, so $x \leqslant -1$, showing that $\bar{\lambda} = x + y\bar{\phi} < -1$. If $y \leqslant -1$, then we have $y\phi < -1$, so $x \geqslant 2$, showing that $\bar{\lambda} = x + y\bar{\phi} > 2$. This proves that $[1, \phi]$ does not contain an element $\lambda \neq 0$ such that $|\lambda| < 1, |\bar{\lambda}| < 1$.

Now suppose the Z-module $[1, \phi]$ does not contain an element $\lambda \neq 0$ satisfying (4.4). We can choose ϕ so that $-1 < \bar{\phi} < 0$, in which case

$\phi = \bar{\phi} + \dfrac{\sqrt{D}}{a} > -1$. Hence, as ϕ cannot satisfy (4.4), we must have $\phi > 1$, so I is reduced.

LEMMA 4. *If* $I = d\left[a, \dfrac{b+\sqrt{D}}{2}\right]$ *is an ideal of* O_D *with* $0 < a < \dfrac{\sqrt{D}}{2}$ *then* I *is reduced.*

Proof. We can write $I = da[1, \phi]$ with $-1 < \bar{\phi} < 0$. Then we have $\phi = \bar{\phi} + \dfrac{\sqrt{D}}{a} > 1$ so that I is reduced.

5. LAGRANGE'S REDUCTION PROCEDURE

In this section we describe Lagrange's reduction procedure which was first introduced in [2]. This procedure uses Lagrange neighbours and so is based on the continued fraction algorithm. The procedure, when applied to a given primitive ideal I of O_D, gives all the reduced ideals of O_D which are equivalent to I.

Let $\{a, b\}$ be a representation of the primitive ideal I of O_D. The Lagrange neighbour of $\{a, b\}$ is the representation $\{a', b'\}$ of the primitive ideal I' of O_D given as follows:

$$(5.1)\quad \begin{cases} q = [\phi] = \left[\dfrac{b+\sqrt{D}}{2a}\right], & \phi = q + \dfrac{1}{\phi'}, \\ b' = -b + 2aq, & a' = \dfrac{D - b'^2}{4a} = \dfrac{D-b^2}{4a} + bq - aq^2, \end{cases}$$

(see (2.10) and (2.11)). We write $\{a, b\} \xrightarrow{L} \{a', b'\}$. The primitive ideal $I' = a'[1, \phi']$ is also called the Lagrange neighbour of I.

We note that

$$\phi' = \frac{1}{\phi - q} > 1, [\phi'] \geqslant 1,$$

as $q = [\phi]$. We also remark that if a is kept fixed and ϕ is changed modulo 1 then ϕ', b' and a' do not change. Hence the Lagrange neighbour of $\{a, b\}$ depends only upon the sign of a. If $\{a, b\} \xrightarrow{L} \{a', b'\}$ then by Corollary 1 the