0. Review of connections and hypercohomology

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MANIN'S PROOF OF THE MORDELL CONJECTURE OVER FUNCTION FIELDS

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In the process of translating Manin's proof of Mordell's conjecture over function fields into modern language we found a gap. The arguments in [M] do not suffice to prove Manin's Theorem of the Kernel. We were able to fill this gap by using those arguments to prove a weaker theorem (Theorem 1.4.3 below) and combining this with the function field analogue of Siegel's Theorem and Manin's ideas to complete the proof of Function Field Mordell. More recently, Chai [C] (see also the Appendix, below) has applied Deligne's Theorem on the semi-simplicity of the action of the monodromy group to deduce Manin's Theorem of the Kernel as reformulated below from the weaker theorem mentioned above. I believe that all this is testimony to the power and depth of Manin's intuition. We were also able to make Manin's analytic proof completely algebraic. Manin has kindly verified that the corrections discussed herein are necessary and apt (see letter to Izvestia...)

In light of the above and because of the ground braking nature of the work we believe that Manin's paper "Rational Points of Algebraic Curves over Function Fields" merits a clear modern treatment. We attempt to give one below.

I. THE THEOREM OF THE KERNEL

0. REVIEW OF CONNECTIONS AND HYPERCOHOMOLOGY

(See also [D-1].) Let S be smooth connected scheme over a ring K. Let $\mathcal{O}_S$ denote the structure sheaf of S, Ω_S^p the sheaf of p-forms on S over K and d the exterior derivation from Ω_S^p to Ω_S^{p+1}. Let $\mathcal{S}$ be a coherent sheaf on S. A connection on $\mathcal{S}$ over K is a K-linear homomorphism $\nabla : \mathcal{S} \to \Omega_S^1 \otimes \mathcal{S}$ satisfying the Leibnitz rule

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$$\nabla(fs) = df \otimes s + f\nabla(s) .$$

for f a local section of $\mathcal{O}_S$ and s a local section of $\mathcal{S}$. We will also say that $(\mathcal{S}, \nabla)$ is a connection on S. There is a K-linear map which we also denote by ∇ from $\Omega_S^p \otimes \mathcal{S} \to \Omega_S^{p+1} \otimes \mathcal{S}$ characterized by

$$\nabla(\omega \otimes s) = d\omega \otimes s + (-1)^p \omega \otimes \nabla(s)$$

for ω a local section of Ω_S^1 and s a local section of $\mathcal{S}$. We say that $(\mathcal{S}, \nabla)$ is integrable if the map $\nabla \circ \nabla : \mathcal{S} \to \Omega_S^2 \otimes \mathcal{S}$ is zero. In this case

$$\mathcal{S} \xrightarrow{\nabla} \Omega_S^1 \otimes \mathcal{S} \xrightarrow{\nabla} \Omega_S^2 \otimes \mathcal{S} \xrightarrow{\nabla} \ldots$$

is a complex. We let $H^i(\mathcal{S}, \nabla)$ denote the i-th hypercohomology group of this complex. When K is a field of characteristic zero, integrability also implies that $\mathcal{S}$ is locally free.

If (H, ∇_H) and (G, ∇_G) are two connections on S then there are natural connections $\nabla_H \otimes \nabla_G$ on $H \otimes G$ and $\nabla_{H,G}$ on $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$ characterized by the formulas

$$\nabla_H \otimes \nabla_G(h \otimes g) = \nabla_H(h) \otimes g + h \otimes \nabla_G(g)$$
$$\nabla_{H,G}(r)(h) = \nabla_G(r(h)) - r(\nabla_H(h))$$

for local sections h and g of H and G and a local section r of $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$. We let $\check{H} = \mathrm{Hom}(H, \mathcal{O}_S)$ and $\check{\nabla}_H = \nabla_{H, \mathcal{O}_S}$, which is a connection on $\check{H}$. It is easy to see that $\nabla_G \otimes \check{\nabla}_H$ equals $\nabla_{H,G}$ under the natural identification of $\mathrm{Hom}_{\mathcal{O}_S}(H, G)$ with $G \otimes \check{H}$. We will need the following, easy to check, lemma.

LEMMA 1.0.1. *Suppose* $r \in \mathrm{Hom}_{\mathcal{O}_S}(H, \Omega_S^p \otimes G) \cong \Omega_S^p \otimes \mathrm{Hom}_{\mathcal{O}_S}(H, G)$. *Then*

$$\nabla_{H,G}(r)(s) = \nabla_G(r(s)) + (-1)^p r(\nabla_H(s)) .$$

Since we will use it frequently in the following we will record here the Čech definition of hypercohomology. (See also [H-1, Chapter 1 § 3].) Suppose $(\mathcal{S}^\bullet, d)$ is a bounded below complex of Abelian sheaves on a topological space S. Then we define the hypercohomology of $\mathcal{S}$ as follows: First let $\mathcal{U}$ be an ordered open cover of S. We have the Čech complexes

$$C^i(\mathcal{U}, \mathcal{S}^j) = \oplus\, \mathcal{S}^j(U)$$

where the sum runs over all intersections U of $i+1$ distinct elements of $\mathscr{U}$. Let $\check{\delta}: C^i(\mathscr{U}, \mathscr{S}^j) \to C^{i+1}(\mathscr{U}, \mathscr{S}^j)$ be the Čech co-boundary. We also have boundaries $d: C^i(\mathscr{U}, \mathscr{S}^j) \to C^i(\mathscr{U}, \mathscr{S}^{j+1})$.

Now let

$$C^n(\mathscr{U}, \mathscr{S}^\cdot) = \oplus\, C^p(\mathscr{U}, \mathscr{S}^q)$$

where the sum runs over $p+q=n$. For $c \in C^n(\mathscr{U}, \mathscr{S}^\cdot)$, we let $c^{p,q}$ denote its p, q-th component. The hyper-coboundary

$$\partial: C^n(\mathscr{U}, \mathscr{S}^\cdot) \to C^{n+1}(\mathscr{U}, \mathscr{S}^\cdot)$$

is defined as follows: For $c \in C^n(\mathscr{U}, \mathscr{S}^\cdot)$, we set

$$(\partial c)^{p,q} = dc^{p-1,q} + (-1)^{p-1}\check{\delta} c^{p,q-1}\,.$$

Then the hypercohomology of $\mathscr{S}$ with respect to $\mathscr{C}$, $\mathbf{H}^\cdot(S, \mathscr{S}^\cdot, \mathscr{C})$, is defined to be Ker$(\partial)$/Image$(\partial)$ and $\mathbf{H}^\cdot(S, \mathscr{S}^\cdot)$ is defined to be an appropriate limit of these groups over all ordered covers. In particular, if S is a scheme, $\mathscr{S}^\cdot$ is a complex of coherent sheaves and $\mathscr{C}$ is an affine open cover, then $\mathbf{H}^\cdot(S, \mathscr{S}^\cdot)$ is naturally isomorphic to $\mathbf{H}^\cdot(S, \mathscr{S}^\cdot, \mathscr{C})$. If in addition S is affine $\mathbf{H}^\cdot(S, \mathscr{S}^\cdot) \cong H^\cdot(\Gamma(\mathscr{S}^\cdot))$.

1. Extensions of connections

Let S be smooth connected scheme over a field K of characteristic zero. Suppose (H, ∇_H) and (G, ∇_G) are integrable connections on S. The set of isomorphism classes of integrable extensions of (H, ∇_H) by (G, ∇_G) forms a group under Baer sum which we will call Ext(H, G).

Proposition 1.1.1. Ext$(H, G) \cong H^1(G \otimes \check{H}, \nabla_G \otimes \check{\nabla}_H)$.

Proof. Since ∇_H is integrable, H is locally free. Let $\mathscr{C}$ be an ordered affine open cover of S such that $H(U)$ is a free $\mathscr{O}_S(U)$-module for each $U \in \mathscr{C}$. Suppose we have an extension

$$0 \to (G, \nabla_G) \to (E, \nabla) \to (H, \nabla_H) \to 0$$

of connections. Let $U \in \mathscr{C}$. Since $H(U)$ is free, there exists an $\mathscr{O}_S(U)$-module section $s_U: H(U) \to E(U)$. Now let $h_U = \nabla \circ s_U - s_U \circ \nabla_H$. We claim that h_U is an $\mathscr{O}_S(U)$-module homomorphism from $H(U)$ into $\Omega^1_S \otimes G(U)$, i.e. an element of $\mathrm{Hom}_{\mathscr{O}_S}(H, \Omega^1_S \otimes G)(U)$. Indeed, for $f \in \mathscr{O}_S(U)$ and $v \in H(U)$,