5. The algebraic proof

Proof. Let s and t be elements of $A(S)$. Define the map $g: A \to A$ by $g = m \circ (id, t \circ f)\,(g(x) = x + t(f(x)))$. Then $g^*: H^1_{DR}(A/S) \to H^1_{DR}(A/S)$ is the identity so that $g^*M(e, s) = M(e, s)$ on the one hand and $g^*M(e, s) = M(t, s+t)$ by Proposition 1.3.2 on the other. Hence,

$$M(s) + M(t) = M(e, s) + M(e, t) = M(t, s + t) + M(e, t) = M(e, s + t)$$

by Proposition 1.3.1. □

Let (B, τ) denote the $K(S)/K$ trace of $A_{K(S)}$ (see [L-AV]). In particular, B is an Abelian scheme over K and $\tau: B \times \mathrm{spec}(K(S)) \to A_{K(S)}$ is a homomorphism. Since K has characteristic zero τ is a closed immersion. Philosophically, B is the largest constant Abelian subscheme of $A_{K(S)}$ defined over over K. The morphism τ extends uniquely to an S-morphism $\bar{\tau}: B \times_K S \to A$. It follows that $B(K)$ maps naturally into $A(S)$. We call the elements s of $A(S)$ such that ns is in the image of $B(K)$, the constant sections of A/S.

PROPOSITION 1.4.2. *The kernel of* M *contains all constant sections of* A/S.

Proof. Let s be a constant section of A/S. Then there exists a positive integer n such that $ns = \bar{\tau} \circ (t \times id)$ where $t \in B(K)$. Hence it follows from the above theorem, Proposition 1.3.2 and Proposition 1.3.4 that $nM(s) = M(ns) = M(\bar{\tau}(t \times id)) = \bar{\tau}^*M(t \times id) = 0$. Since

$$\mathrm{Ext}(H^1_{DR}(A/S), K[S])$$

is uniquely divisible, by Corollary 1.1.2, the proposition follows. □

We wish to prove the conserve of this proposition. I.e. we wish to prove:

THEOREM 1.4.3. *The kernel of* M *is precisely the group of all constant sections of* A/S.

We will give two proofs of this result. The first is Algebraic. The second is analytic and is essentially a reformulation of Manin's proof based on remarks by Katz [K2] in a letter to Ogus.

5. THE ALGEBRAIC PROOF

a. *Differentials with logarithmic singularities*

(See [K] § 1.0). Suppose X is a smooth scheme over a scheme T and Z is a hypersurface in X whose irreducible components are smooth over T and cross normally relative to T. Let $W = X - Z$ and $\tilde{Z}$ the disjoint union of the irreducible components of Z. Let $(\Omega^{\cdot}_{X/T}(\mathrm{Log}(Z)), d)$ denote the complex of differentials on X/T with logarithmic singularities along Z. (When $T = K$, we drop T from the notation.) When T has characteristic zero, which we will now assume, the i-th hypercohomology group of this complex is naturally isomorphic to $H^i_{DR}(W/T)$. We have a natural short exact sequence of complexes

$$0 \to \Omega^{\cdot}_{X/T} \to \Omega^{\cdot}_{X/T}(\mathrm{Log}(Z)) \overset{\mathrm{Res}}{\to} \Omega^{\cdot}_{\tilde{Z}/T}(-1) \to 0$$

From which, upon taking cohomology, we obtain the long exact sequence:

$$(5.1) \qquad 0 \to H^1_{DR}(X/T) \to H^1_{DR}(W/T) \to H^0_{DR}(\tilde{Z}/T) \to H^2_{DR}(X/T) \\ \to H^2_{DR}(W/T) \to H^2_{DR}(\tilde{Z}/T)$$

In addition, we have a short exact sequence of complexes

$$0 \to \Omega^1_S \otimes \Omega^{\cdot}_{X/S}(\mathrm{Log}(Z))(-1) \to \Omega^{\cdot}_X(\mathrm{Log}(Z)) \to \Omega^{\cdot}_{X/S}(\mathrm{Log}(Z)) \to 0 .$$

The boundary maps in the long exact sequence of hypercohomology obtained from this short exact sequence are the Gauss-Manin connections $\nabla: H^i_{DR}(W/S) \to \Omega^1_S \otimes H^i_{DR}(W/S)$. Moreover the long exact sequence (5.1) is horizontal with respect to all the Gauss-Manin connections.

If D is any divisor on X, let $\eta_T(D)$ denote the cohomology class of D in $H^2_{DR}(X/T)$. Recall ([H-DR; 7.7]), if $\mathscr{L}$ is an ordered affine open cover of C and $\{f_U\}$ is a Čech one-cochain with with coefficients in $\mathscr{O}_A$ with respect to $\mathscr{L}$ such that the divisor of f_U is the restriction of D to U, then $\eta_T(D)$ is the cohomology class represented by the hyper one-cocycle $(0, \{d_{C/T}\mathrm{Log}(f_{U,V})\}, 0)$, where $f_{U,V} = f_U/f_V (U < V)$. Suppose now that T is affine. Then $H^0_{DR}(\tilde{Z}/T)$ is naturally isomorphic to the group of divisors on X supported on Z with coefficients in $K[T]$.

LEMMA 1.5.1. *Suppose D is a divisor on X supported on Z, then the image of D in $H^2_{DR}(X/T)$ via the appropriate map in (5.1) is equal to $\eta_T(D)$.*

Proof. This is essentially Proposition 7.6 of [H]. We carry out the proof in order to "straighten out" the sign.

Let $\mathscr{L}$ be an affine open cover of X and $\{f_U\}$ is a Čech one-chain with coefficients in $\mathscr{O}_X$ with respect to $\mathscr{L}$ such that the divisor of f_U is the restriction of D to U. Then, $d\mathrm{Log}(f_U) \in \Omega^1_{X/T}(\mathrm{Log}(Z))(U)$ and $\mathrm{Res}(d\mathrm{Log}(f_U))$ is the image of the image of D in $H^0_{DR}(\tilde{Z}/T) \cong \mathscr{O}_{\tilde{Z}}(U)$. It follows that the image of D in $H^2_{DR}(X/T)$ is the class of the hyper-coboundary of $(\{d\text{Log}(f_U)\},0)$ which is $\eta_T(D)$ by definition. □

By a properly semi-stable curve over S, we mean a curve over S such that the irreducible components of the closed fibers are smooth and cross normally. (The irreducible components do not have to be smooth if the curve is only semi-stable.)

COROLLARY 1.5.2. *Suppose R is a smooth connected curve over a field K and X is a properly semi-stable curve over R smooth over K. Suppose U is a non-empty open subset of R and $Y = R - U$. Then the kernel of the natural map from $H^2_{DR}(X)$ into $H^2_{DR}(X_U)$ is generated by $\{\eta(D)\}$ where D runs over the irreducible components of X_Y.*

Proof. This follows from the lemma and the exact sequence (5.1), since the closed fibers of C/T are unions of smooth hypersurfaces of C which cross normally. □

LEMMA 1.5.3. *With notation as in the above corollary, if R is affine and X is smooth over R then the map from $H^2_{DR}(X) \to H^2_{DR}(X_U)$ is an injection.*

Proof. For a closed point x of R, let X_x denote the fiber above x. Since all the fibers of X over S are smooth, it follow from the corollary that the the kernel of the map $H^2_{DR}(X) \to H^2_{DR}(X_U)$ is generated by $\{\eta(X_x)\}$ where x runs over the closed points of Y. Now $\eta(X_x)$ is the pull-back of $\eta(x) \in H^2_{DR}(R)$. As this latter group is zero, this proves the lemma. □

b. *End of algebraic proof*

First by using the functoriality of M, Proposition 1.3.2, and the fact that every Abelian variety over S is the quotient of a Jacobian over S we may assume that A is the Jacobian of a smooth proper curve C over S. By Proposition 1.1.1 and the long exact sequence (2.3), $\text{Ext}(H^1_{DR}(C/S)^\vee, K[S])$ maps naturally into $H^2_{DR}(C)$. Moreover, since C is a proper smooth connected curve over $S, H^1_{DR}(C/S)$ is canonically isomorphic to $H^1_{DR}(C/S)^\vee$. The fact we need to finish the proof is:

PROPOSITION 1.5.4. *Let s and t be two elements of $C(S)$. The class $\eta(t-s)$ is equal to the image of $M(s,t)$ in $H^2_{DR}(C)$.*

By the previous lemma and the functoriality of η we may shrink S to suppose that $s \cap t = \varnothing$. To prove the proposition, we need the next lemma.

Suppose now that $T = S, Z = s \cup t$ and $X = C$. Then the exact sequence (5.1) becomes:

$$(5.2) \qquad 0 \to H^1_{DR}(C/S) \to H^1_{DR}(W/S) \to H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S) \to 0$$

Furthermore $H^2_{DR}(C/S)$ is canonically isomorphic to $K[S]$ with generator $\eta_S(s) = \eta_S(t)$ and so the kernel of $H^0_{DR}(\tilde{Z}/S) \to H^2_{DR}(C/S)$ is a principal $K[S]$ module with generator $D = s - t$. Using this generator, (5.2) yields an extension $B_{s,t}$ of the connection $(K[S], d)$ by $(H^1_{DR}(C/S), \nabla)$.

LEMMA 1.5.5. *Identifying* $H^1_{DR}(C/S)$ *with* $H^1_{DR}(C/S)^\vee$, *the extension* $B_{s,t}$ *is isomorphic to the dual of* $E_{s,t}$.

Proof. Regarding the complexes $\Omega^{\cdot}_{C/S,Z}$ and $\Omega^{\cdot}_{C/S}(\mathrm{Log}(Z))$ as subcomplexes of $\Omega^{\cdot}_{W/S,Z}$ the wedge product gives a product from

$$\Omega^{\cdot}_{C/S.Z} \times \Omega^{\cdot}_{C/S}(\mathrm{Log}(Z))$$

into $\Omega^{\cdot}_{C/S}$ which induces a pairing

$$(\ ,\ ): H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S) \to H^2_{DR}(C/S) \cong K[S]\ .$$

This pairing is compatible with the exact sequences

$$0 \to H^0(C, \Omega^1_{C/S}) \to H^1_{DR}(C/S, Z) \to H^1(C, \Omega^0_{X/S}) \to 0$$

$$0 \to H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)) \to H^1_{DR}(W/S) \to H^1(C, \mathscr{O}_C) \to 0$$

arising from the Hodge to de Rham spectral sequences for hypercohomology (which degenerate). In other words, the image of $H^0(C, \Omega^1_{C/S})$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z))$ in $H^1_{DR}(W/S)$ and if we identify $\Omega^0_{X,Z}$ with $\mathscr{O}_C(Z)$ and $\Omega^1_{C/S}(\mathrm{Log}(Z))$ with $\Omega^1_{C/S}(-Z)$ the pairings induced on $H^0(C, \Omega^1_{C/S}) \times H^1(C, \mathscr{O}_C)$ and on

$$H^1(C, \Omega^0_{X,Z}) \times H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)))$$

are the natural ones. Since these pairings are non-degenerate, it follows that the pairing on $H^1_{DR}(C/S, Z) \times H^1_{DR}(W/S)$ is non-degenerate.

It is also clear that the image of $H^0_{DR}(Z/S) \cong K[Z]$ in $H^1_{DR}(C/S, Z)$ is perpendicular to the image of $H^1_{DR}(C/S)$ in $H^1_{DR}(W/S)$ and that the pairing induced on $H^1_{DR}(C/S) \times H^1_{DR}(C/S)$ is the natural one.

The lemma will follow from the following claim: Let ι denote the map from $K[Z]$ to $H^1_{DR}(C/S, Z)$ and Res the map from $H^1_{DR}(W/S)$ to $K[Z]$. Let $T_{Z/S}$ denote the trace from $K[Z]$ to $K[S]$. Suppose $c \in K[Z]$ and $\omega \in H^1_{DR}(W/S)$. Then

$$(\iota(c), w) = -T_{Z/S}(c\,\mathrm{Res}(\omega)).$$

Indeed, if $s^*c = 0$, $t^*c = 1$, $\mathrm{Res}_s(\omega) = 1$ and $\mathrm{Res}_t(\omega) = -1$ then $-T_{Z/S}(c\,\mathrm{Res}(\omega)) = 1$.

To prove this claim we may shrink S. Hence, we may assume first that $\{U, V\}(U < V)$ is an ordered affine open cover of C such that $U = C - s$ and $V = C - t$, second that $\iota(c)$ is represented by a hypercocycle of the form $\partial(\{g_U, g_V\})$ where $s^*g_U = s^*c$ and $t^*g_V = t^*c$ and third, since the composition $H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)) \to H^1_{DR}(W/S) \to K[Z]$ is surjective, that w is in the image of $H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)))$, i.e., w is represented by a hypercocycle of the form $(\{\omega_U, \omega_V\}, 0)$ where $\omega_U = \omega = \omega_V$ on $U \cap V$ for some $\omega \in H^0(C, \Omega^1_{C/S}(\mathrm{Log}(Z)))$. It follows that $(\iota(c), w)$ as an element of $H^1_{DR}(C, \Omega^1_{C/S}) \cong H^2_{DR}(C/S)$ is represented by the cocycle $\{\nu_{U,V}\}$ with $\nu_{U,V} = (g_V - g_U)\omega$. Since the image of this element in $K[S]$ is

$$\begin{aligned}\mathrm{Res}_s(-g_U\omega) + -\mathrm{Res}_t(g_V\omega) &= -(s^*g_U\mathrm{Res}_s(\omega) + t^*g_V\mathrm{Res}_t(\omega)) \\ &= -T_{T/S}(c\,\mathrm{Res}(\omega)).\end{aligned}$$

this establishes the claim and the lemma. □

End of proof of Proposition 1.5.4

Consider the commutative diagram of complexes of sheaves with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{C/S}(-1) & \to & \Omega^{\cdot}_C & \to & \Omega^{\cdot}_{C/S} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{C/S}(\mathrm{Log}(Z))(-1) & \to & \Omega^{\cdot}_C(\mathrm{Log}(C)) & \to & \Omega^{\cdot}_{C/S}(\mathrm{Log}(Z) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_S \otimes \Omega^{\cdot}_{Z/S}(-2) & \to & \Omega^{\cdot}_Z(-1) & \to & \Omega^{\cdot}_{Z/S}(-1) & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0\,. & &
\end{array}$$

If we take hyper-cohomology of this diagram we obtain a commutative diagram

$$\begin{array}{ccccccc} & & & & H^1_{DR}(C/S) & \overset{\nabla}{\to} & \Omega^1_S \otimes H^1_{DR}(C/S) \\ & & & & \downarrow & & \downarrow \\ & & & & H^1_{DR}(W/S) & \overset{\nabla}{\to} & \Omega^1_S \otimes H^1_{DR}(W/S) \\ & & & & \downarrow & & \\ & & H^0_{DR}(Z) & \to & H^0_{DR}(Z/S) & & \\ & & \downarrow & & \downarrow & & \\ H^1_{DR}(C/S) \overset{\nabla}{\to} \Omega^1_S \otimes H^1_{DR}(C/S) & \to & H^2_{DR}(C) & \to & H^2_{DR}(C/S) & & \end{array}$$

with exact rows and columns in which the bottom row is part of the Leray long exact sequence. Let a be the element in $H^0_{DR}(Z)$ corresponding to the divisor $s - t$. The image of a in $H^2_{DR}(C)$ is $\eta(s - t)$ by Lemma 1.5.1. On the other hand the image of a in $H^0_{DR}(Z/S)$ is our chosen generator of the kernel of the map to $H^2_{DR}(C/S)$. In particular, it is the image of an element b of $H^1_{DR}(W/S)$ and $\nabla(b)$ is the image of an element c of $\Omega^1_S \otimes H^1_{DR}(C/S)$ whose image in $H^2_{DR}(C/S)$ is the same as that of a by an elementary diagram chase. On the other hand, the image of c in $H^1(H^1_{DR}(C/S), \nabla)$ is the class corresponding to the extension $B_{s,t}$ by definition (see Proposition 1.1.1) which is, after identifying $H^1_{DR}(C/S)$ with $H^1_{DR}(C/S)^\vee$, $-M(s, t)$ by Lemma 1.5.1 and Lemma 1.5.3. Hence the image of $M(s, t)$ in in $H^2_{DR}(C)$ is $-\eta(s - t) = \eta(t - s)$ as required. □

Now we are in a position to prove the Theorem 1.4.3. We will suppose $M(s, t) = 0$ which amounts to $\eta_S(t - s) = 0$ by the Proposition 1.5.1. Recall, that A is the Jacobian of C/S. Let d denote the divisor class of $t - s$ in $A(K(C))$. We will show that the canonical height of d is zero. We may replace S by a finite étale cover and complete C to a properly semi-stable curve $\tilde{C}$ over the completion $\tilde{S}$ of S which is smooth over K. Let D be a $\mathbf{Q}$-rational divisor on $\tilde{C}$ which is perpendicular (under the intersection pairing) to all the irreducible components of all the fibers of $\tilde{C}/\tilde{S}$ and whose restriction to C is $t - s$. Such a divisor exists by the function field analogue of Theorem 1.3 of [Hr] (see also Theorem 5.1 (i) of [Ch]). It follows that the image of $\eta(D)$ in $H^2_{DR}(C)$ is $\eta(t - s) = 0$. Corollary 1.5.2 implies that $\eta(D)$ is in the span of $\{\eta(Y)\}$ where Y runs over the irreducible component of the closed fibers above $\tilde{C} - C$. In particular, $D \cdot D = 0$ using Theorem 7.8.2 of [H]. On the other hand, $D \cdot D$ is -2 times the canonical height of d by the function field analogue of Theorem 5.1 of [Ch]. It now follows from Theorem 5.4.1 of [L], that the image of $t-s$ in $J(C)$ is a constant section which completes the proof. $\square$

6. The Analytic Proof

In this section we will suppose $K = \mathbf{C}$.

a. *The Poincaré Lemma*

Suppose $(\mathscr{S}, \nabla)$ is a sheaf on S^{an} with integrable connection. Then by the Poincaré lemma for integrable connections, it follows that the complex of sheaves

$$\mathscr{S} \xrightarrow{\nabla} \Omega^1_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \Omega^2_{S^{an}} \otimes \mathscr{S} \xrightarrow{\nabla} \cdots$$

is a resolution of the sheaf $\mathscr{S}^{\nabla}$. Hence,

Proposition 1.6.1. *$H^i(\mathscr{S}, \nabla)$ is naturally isomorphic to $H^i(S, \mathscr{S}^{\nabla})$.*

Remark. As in Proposition 1.1.1, $H^1(\mathscr{S}, \nabla)$ is isomorphic to $\mathrm{Ext}(\mathscr{S}^{\vee}, \mathscr{O}_{S^{an}})$. We can describe the isomorphism from $H^1(\mathscr{S}, \nabla)$ to $H^1(S, \mathscr{S}^{\nabla})$ explicitly as follows: Let h be an element of $H^1(\mathscr{S}, \nabla)$. Let $\mathscr{C}$ is a covering of S by open disks. Suppose $\mathscr{E}$ is an extension of $\mathscr{S}^{\vee}$ by $\mathscr{O}_{S^{an}}$ corresponding to h. Then $\mathscr{E}^{\vee}$ is an extension of $\mathscr{O}_{S^{an}}$ by $\mathscr{S}$. For each $U \in \mathscr{C}$, there exists an $s_U \in \mathscr{E}^{\vee}(U)^{\nabla}$ which maps to 1 in $\mathscr{O}_{S^{an}}(U)$. Then the image h in $H^1(S, \mathscr{S}^{\nabla})$ is the class of the cocycle $\{(U, V) \to s_U - s_V\}$.

Suppose, X is a smooth proper S-scheme and Z is a subscheme of X which is either empty or finite over S. We will define the Betti homology sheaf $\mathscr{H}_1(X/S, Z, \mathbf{Z})$ on S^{an} as follows. If Z is smooth over S, we define $\mathscr{H}_i(X/S, Z, \mathbf{Z})$ to be the sheaf associated to the presheaf

$$U \to H_i(f^{-1}(U), f^{-1}(U) \cap Z, \mathbf{Z}) ,$$

(this latter group is the Betti homology of $f^{-1}(U)$ relative to $f^{-1}(U) \cap Z$). More generally, let S' be a non-empty affine open subset of S such that $Z' = Z \times_S S'$ is étale over S'. Let $X' = X \times_S S'$ and let ι denote the inclusion morphisms $X' \to X$, $Z' \to Z$ and $S' \to S$. We set

$$\mathscr{H}_i(X/S, Z, \mathbf{Z}) = \iota_* \mathscr{H}_i(X'/S', Z', \mathbf{Z}) .$$

This is independent of the choice of S'. We also set

$$\mathscr{H}_i(X/S, \mathbf{Z}) = \mathscr{H}_i(X/S, \varnothing, \mathbf{Z}) \text{ and } \mathscr{H}_1(X/S, Z, \mathbf{C}) = \mathscr{H}_1(X/S, Z, \mathbf{Z}) \otimes \underline{\mathbf{C}} .$$