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2. PICARD-FUCHS COMPUTATIONS

We will need an explicit formula for $\mu(s, t)$ in some cases. Suppose that X/S has relative dimension one. Suppose $z \in K[S]$ such that $\Omega_S^1(S) = K[S]dz$ and suppose U is an affine open of $X, s \in U(S)$ and $v \in \mathcal{O}_X(U)$, such that $s^*v = 0$ and $\Omega_{X/S}^1(U) = \mathcal{O}_X(U)d_{X/S}v$. For $u \in \mathcal{O}_X(U)$ we define $\partial_z u$ and $\partial_v u$ by

$$du = \partial_z u dz + \partial_v u dv$$

Clearly ∂_z is a lifting of $\partial = : \partial / \partial z$ to a derivation of $\mathcal{O}_X(U)$. For $\omega = ud_{X/S}v \in \Omega_{X/S}^1(U)$ we set $\partial_z \omega = \partial_z u d_{X/S}v$ (the image of the Lie derivative of udv with respect to ∂_z in $\Omega_{X/S}^1(U)$). Since ∂ generates \mathcal{D} over $K[S]$ we can and will also make \mathcal{D} act on $\Omega_{X/S}^1(U)$ using ∂_z .

LEMMA 2.2.1. *Suppose $\omega = ud_{X/S}v \in \Omega_{X/S}^1(U)$ is of the second kind and $[\omega]$ is its class in $H_{DR}^1(X/S)$. Then*

$$\partial[\omega] = [\partial_z \omega] .$$

Proof. The element udv is a lifting of $ud_{X/S}v$ to $\Omega_X^1(U)$, and $d(udv) = du \wedge dv = \partial_z u dz \wedge dv$. Since this is the image of $dz \otimes \partial_z \omega$ in Ω_X^2 the lemma follows. \square

COROLLARY 2.2.2. *Suppose $\sum D_i \otimes \omega_i \in PF$. Then*

$$\sum D_i \omega_i = d_{X/S} w$$

for some $w \in \mathcal{O}_X(U)$.

Suppose $t \neq s$ is an element of $U(S)$ and $Z = s \cup t$. Let l denote the map from $K[S]$ into $H_{DR}^1(U/S, Z)$ associated to the pair (s, t) . For $\omega \in \Omega_{X/S}^1(U)$ let $[\omega]_Z$ denote the class of ω in $H_{DR}^1(U/S, Z)$.

LEMMA 2.2.3. *Suppose U, s and v are as above, $t \in U(S)$ and $t^*v \neq 0$. Suppose $\omega = ud_{X/S}v \in \Omega_{X/S}^1(U)$. Then $\partial^k[\omega]_Z$ equals*

$$[\partial_z^k \omega]_Z + l(\sum \partial^{i-1}(t^*(\partial_z^{k-i} u) \partial t^* v))$$

where i runs from 1 to k .

Proof. By shrinking S we may suppose that t^*v is invertible. We want to compute $\nabla[\omega]_Z$. First we must lift $ud_{X/S}v$ to section of $\Omega_{X,Z}^1(U)$. Let $y = f^*(t^*v)$. Then $\eta = u y dy^{-1} v$ is such a lifting and it equals $udv - u y y^{-1} \partial_z y dz$. Then $\nabla[\omega]_Z$ is the class of

$$d\eta = \partial_z u dz \wedge dv - d(uvy^{-1}) \wedge dy = dz \wedge \partial_z u dv + dz \wedge d(uvy^{-1} \partial_z y).$$

which is the image of

$$dz \otimes (\partial_z \omega + d_{X/S}(uvy^{-1} \partial_z y)) \in \Omega_S^1 \otimes \Omega_{X/S}^1(U).$$

Hence $\partial[\omega]$ is the class of $\partial_z \omega + d_{X/S}(uvy^{-1} \partial_z y)$ in $H_{DR}^1(U/S, Z)$. Since $(t^* - s^*)(uvy^{-1} \partial_z y) = t^* u \partial(t^* v)$ the lemma follows in the case $k = 1$. Since $\partial \circ l = l \circ \partial$ the lemma follows in general by induction. \square

COROLLARY 2.2.4. *Suppose U, s, z and v are as above, $t \in X(S)$ which meets U and $t^* v \neq 0$. Suppose ω, ω' and ω'' are elements $\omega_{X/S}$. Let $\omega = u d_{X/S} v$ and $\omega' = u' d_{X/S} v$ on U . Then we have:*

(i) *Suppose $\mu = \partial \otimes \omega - 1 \otimes \omega' \in PF$, $\omega = u d_{X/S} v$ and $\partial_z \omega - \omega' = d_{X/S} w$, with $w \in \mathcal{O}_X(U)$. Then*

$$\mu(s, t) = t^* w - s^* w + (t^* u) \partial t^* v.$$

(ii) *Suppose $\mu = \partial^2 \otimes \omega + \partial \otimes \omega' + 1 \otimes \omega'' \in PF$ and $\partial^2 \omega + \partial \omega' + \omega'' = d_{X/S} w$ with $w \in \mathcal{O}_X(U)$. Then*

$$\mu(s, t) = t^* ((w - s^* w, (u' + 2\partial_z u), \partial_v u, u) \cdot (1, x_t, x_t^2, \partial x_t))$$

and where $x_t = \partial t^* v$.

Proof. First shrink S so that s and t satisfy the hypotheses of the lemma and then apply it and the definition of $\mu(s, t)$. \square

Suppose $g: X \rightarrow A$ is a morphism over S from a curve to an Abelian scheme. Suppose $\kappa_{A/S}$ is an isomorphism. If $\eta = g^* \omega$ where $\omega \in \omega_{A/S}$ we will set $\mu_\eta = g^* \mu_\omega$. This is independent of the choice of ω . As an immediate consequence of the previous corollary we obtain:

COROLLARY 2.2.5. *Let U, z, s and v be as above. Set $X(S)' = \{t \in X(S) : t \text{ meets } U \text{ and } t^* v \neq 0\}$. Then there exist maps*

$$V = : V_{z, v} : T_{U, v} \rightarrow K(S)^4$$

and

$$L = : L_{z, v, s} : \omega_{X/S} \rightarrow K(X)^4$$

such that L is K -linear and for $t \in X(S)'$ and $\omega \in g^* \omega_{A/S}$,

$$\mu_\omega(s, t) = t^*(L(\omega) \cdot V(t)).$$