2. PICARD-FUCHS COMPUTATIONS

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

2. Picard-Fuchs computations

We will need an explicit formula for $\mu(s,t)$ in some cases. Suppose that X/S has relative dimension one. Suppose $z \in K[S]$ such that $\Omega^1_S(S) = K[S]dz$ and suppose U is an affine open of X, $s \in U(S)$ and $v \in \mathscr{O}_X(U)$, such that $s^*v = 0$ and $\Omega^1_{X/S}(U) = \mathscr{O}_X(U)d_{X/S}v$. For $u \in \mathscr{O}_X(U)$ we define $\partial_z u$ and $\partial_v u$ by

$$du = \partial_z u dz + \partial_v u dv$$

Clearly ∂_z is a lifting of $\partial =: \partial/\partial z$ to a derivation of $\mathscr{O}_X(U)$. For $\omega = u d_{X/S}v \in \Omega^1_{X/S}(U)$ we set $\partial_z\omega = \partial_z u d_{X/S}v$ (the image of the Lie derivative of udv with respect to ∂_z in $\Omega^1_{X/S}(U)$). Since ∂ generates $\mathscr{D}$ over $K[S]$ we can and will also make $\mathscr{D}$ act on $\Omega^1_{X/S}(U)$ using ∂_z.

Lemma 2.2.1. *Suppose* $\omega = u d_{X/S}v \in \Omega^1_{X/S}(U)$ *is of the second kind and* $[\omega]$ *is its class in* $H^1_{DR}(X/S)$. *Then*

$$\partial[\omega] = [\partial_z\omega] .$$

Proof. The element udv is a lifting of $u d_{X/S}v$ to $\Omega^1_X(U)$, and $d(udv) = du \wedge dv = \partial_z u dz \wedge dv$. Since this is the image of $dz \otimes \partial_z\omega$ in Ω^2_X the lemma follows. □

Corollary 2.2.2. *Suppose* $\sum D_i \otimes \omega_i \in PF$. *Then*

$$\sum D_i\omega_i = d_{X/S}w$$

for some $w \in \mathscr{O}_X(U)$.

Suppose $t \neq s$ is an element of $U(S)$ and $Z = s \cup t$. Let l denote the map from $K[S]$ into $H^1_{DR}(U/S, Z)$ associated to the pair (s,t). For $\omega \in \Omega^1_{X/S}(U)$ let $[\omega]_Z$ denote the class of ω in $H^1_{DR}(U/S, Z)$.

Lemma 2.2.3. *Suppose* U, s *and* v *are as above,* $t \in U(S)$ *and* $t^*v \neq 0$. *Suppose* $\omega = u d_{X/S}v \in \Omega^1_{X/S}(U)$. *Then* $\partial^k[\omega]_Z$ *equals*

$$[\partial_z^k\omega]_Z + l(\textstyle\sum \partial^{i-1}(t^*(\partial_z^{k-i}u)\partial t^*v))$$

where i *runs from* 1 *to* k.

Proof. By shrinking S we may suppose that t^*v is invertible. We want to compute $\nabla[\omega]_Z$. First we must lift $u d_{X/S}v$ to section of $\Omega^1_{X,Z}(U)$. Let $y = f^*(t^*v)$. Then $\eta = uydy^{-1}v$ is such a lifting and it equals $udv - uvy^{-1}\partial_z y dz$. Then $\nabla[\omega]_Z$ is the class of

$$d\eta = \partial_z u dz \wedge dv - d(uvy^{-1}) \wedge dy = dz \wedge \partial_z u dv + dz \wedge d(uvy^{-1}\partial_z y) .$$

which is the image of

$$dz \otimes (\partial_z \omega + d_{X/S}(uvy^{-1}\partial_z y)) \in \Omega^1_S \otimes \Omega^1_{X/S}(U) .$$

Hence $\partial[\omega]$ is the class of $\partial_z\omega + d_{X/S}(uvy^{-1}\partial_z y)$ in $H^1_{DR}(U/S, Z)$. Since $(t^* - s^*)(uvy^{-1}\partial_z y) = t^*u\partial(t^*v)$ the lemma follows in the case $k = 1$. Since $\partial \circ l = l \circ \partial$ the lemma follows in general by induction. □

COROLLARY 2.2.4. *Suppose U, s, z and v are as above, $t \in X(S)$ which meets U and $t^*v \neq 0$. Suppose ω, ω' and ω'' are elements* $\omega_{X/S}$. Let $\omega = ud_{X/S}v$ *and* $\omega' = u'd_{X/S}v$ *on* U. *Then we have:*

(i) *Suppose* $\mu = \partial \otimes \omega - 1 \otimes \omega' \in PF$, $\omega = ud_{X/S}v$ *and* $\partial_z\omega - \omega' = d_{X/S}w$, *with* $w \in \mathscr{O}_X(U)$. *Then*

$$\mu(s, t) = t^*w - s^*w + (t^*u)\partial t^*v .$$

(ii) *Suppose* $\mu = \partial^2 \otimes \omega + \partial \otimes \omega' + 1 \otimes \omega'' \in PF$ *and* $\partial^2\omega + \partial\omega' + \omega'' = d_{X/S}w$ *with* $w \in \mathscr{O}_X(U)$. *Then*

$$\mu(s, t) = t^*((w - s^*w, (u' + 2\partial_z u), \partial_v u, u) \cdot (1, x_t, x_t^2, \partial x_t))$$

and where $x_t = \partial t^* v$.

Proof. First shrink S so that s and t satisfy the hypotheses of the lemma and then apply it and the definition of $\mu(s, t)$. □

Suppose $g: X \to A$ is a morphism over S from a curve to an Abelian scheme. Suppose $\kappa_{A/S}$ is an isomorphism. If $\eta = g^*\omega$ where $\omega \in \omega_{A/S}$ we will set $\mu_\eta = g^*\mu_\omega$. This is independent of the choice of ω. As an immediate consequence of the previous corollary we obtain:

COROLLARY 2.2.5. *Let U, z, s and v be as above. Set* $X(S)' = \{t \in X(S): t$ *meets* U *and* $t^*v \neq 0\}$. *Then there exist maps*

$$V =: V_{z,v}: T_{U,v} \to K(S)^4$$

and

$$L =: L_{z,v,s}: \omega_{X/S} \to K(X)^4$$

such that L is K-linear and for $t \in X(S)'$ and $\omega \in g^*\omega_{A/S}$,

$$\mu_\omega(s, t) = t^*(L(\omega) \cdot V(t)) .$$