3. COROLLARIES OF THE THEOREM OF THE KERNEL

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(iii) $(h_{m,1})^{-1}(T) \cap C_m(S)$ *is infinite,*

(iv) *There exists a finite covering* $S_{m,n}$ *of* S *such that the fiber product of* $h_{m,n}$ *with* $S_{m,n}$ *is Galois, Abelian and of positive degree.*

Let J denote the Jacobian scheme of C over S. Let $a: C \to J$ be an Albanese morphism. Let p be a prime. Let $\bar{T}$ denote the closure of $a(T)$ in $J(S) \otimes \mathbf{Z}_p$. Since $a(T)$ is infinite it follow from the Mordell-Weil Theorem that there exists a $t \in \bar{T} - a(T)$. Let $t_n \in T$ such that $t - a(t_n) \in p^n J(S)$. Let C_n denote the normalization of the fiber-product of C and J via the map $H_n: x \to p^n x + t_n$ and $h_{n,1}$ the natural map from C_n to C. It follows that C_n is defined over S and since $H_m(J(S)) \supseteq \{t_n : m \mid n\}$ that $h_{n,1}(C_m(S))$ contains an infinite subset of T.

All that remains is to exhibit the maps $h_{m,n}$. Clearly, $t_m - t_n = p^n r_{m,n}$ for some $r_{m,n} \in J(S)$. Let $H_{m,n}$ denote the map $x: p^{m-n}x + r_{m,n}$. Then $H_{m,k} = H_{n,k} \circ H_{m,n}$. It follows that $H_{m,n}$ pulls back to a morphism $h_{m,n}: C_m \to C_n$. It is easy to see that this morphism becomes Abelian after adjoining the p^{m-n}-torsion points on J. This proves the proposition. □

Remark. One can also prove the above proposition with the condition $n \leqslant m$ replaced by $n \mid m$.

3. Corollaries of the Theorem of the Kernel

LEMMA 3.3.1. *Suppose* $g: X' \to X$ *is a morphism of smooth proper schemes with geometrically connected fibers over* S. *Then if* $\mu \in PF(X'/S)$ *and* $s, t \in X(S), (g^*\mu)(s, t) = \mu(g \circ s, g \circ t)$.

Proof. This follows easily from Lemma 1.3.2. □

Suppose J is the Jacobian of C over S and g is an Albanese morphism, then since $g^*: H^1_{DR}(J/S) \to H^1_{DR}(C/S)$ is an isomorphism $g^*: PF(J/S) \to PF(C/S)$ is an isomorphism.

LEMMA 3.3.2. *Let* μ *be a fixed Picard-Fuchs differential equation on* C/S. *Then* $\{\mu(s, t): s, t \in C(S)\}$ *lies in a finite dimensional subspace of* $K[S]$ *over* K.

Proof. Suppose $\tilde{\mu} \in PF(J/S)$ such that $g^*\tilde{\mu} = \mu$. The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that $J(S)$ modulo the kernel of the homomorphism $s \to \tilde{\mu}(e, s)$ is a finitely generated Abelian group. □

LEMMA 3.3.3. *Suppose* A *is an Abelian scheme over* S *such that* $[W_{A/S}] = H^1_{DR}(A/S)$ *and* $g: C \to A$ *is a non-constant morphism over* S. *Fix* $s \in C(S)$. *Then the set* $T = \{t \in C(S): (g^*\mu)(s,t) = 0$ *for all* $\mu \in PF(A/S)\}$ *is of bounded height.*

Proof. Let A' denote the smallest Abelian subscheme of A over S containing $g(C)$. Since the map $g^*: PF(A/S) \to PF(A'/S)$ is surjective and $[W_{A/S}] = H^1_{DR}(A/S)$, it follows from Proposition 2.1.2 that $g(T)$ is contained in a translation of the group of constant sections of A'/S. Hence, $g(T)$ is a set of bounded heigt. Finally, since $C \to g(C)$ is a finite morphism, it follows that T is a set of bounded height. □

In particular,

COROLLARY 3.3.4. *Suppose* A *is an Abelian scheme over* S *such that* $\kappa_{A/S}$ *is an isomorphism and* $g: C \to A$ *is a non-constant morphism over* S. *Fix* $s \in C(S)$. *Then the set* $\{t \in C(S): (g^*\mu_\omega)(s,t) = 0$ *for all* $\omega \in \omega_{A/S}\}$ *is of bounded height.*

4. PROOF OF MORDELL'S CONJECTURE

PROPOSITION 3.4.1. *Suppose the kernel of the* $\kappa_{C/S}$ *has rank at least* 2 *over* $K[S]$, *then the points of* $C(S)$ *have bounded height.*

Proof. Suppose $C(S)$ contains points of arbitrarily large height. Fix $s \in C(S)$. By shrinking S, if necessary, we may suppose that there exists a function $z \in K[S]$ such that $\Omega^1_S = K[S]dz$ and there exists a finite covering $\mathscr{L}$ of C by affine opens U and functions $v_U \in \mathscr{O}_C(U)$ such that $s \in U(S)$, and $\Omega^1_C(U)$ is spanned by dz and dv_U. We may also suppose that $s^*v_U = 0$ by replacing v_U with $v_U - (s \circ f)^* v_U$ if necessary. For $U \in \mathscr{L}, u \in \mathscr{O}_C(U)$ we define $\partial_{U,z}u$ and $\partial_{U,v}u$ by the equation

$$du = \partial_{U,z}u dz + \partial_{U,v}u dv_U\ .$$

Then $\partial_{U,z}$ is a lifting of $\partial =: \partial/\partial z$. We set $\mu(t) = \mu(s,t)$ for

$$\mu \in PF =: PF(C/S)$$

and $t \in C(S)$.

Let ω_1 and ω_2 be two independent elements in the kernel of $\kappa_{C/S}$. It follows that there exist ω_1' and $\omega_2' \in \omega_{C/S}$ such that

$$\partial[\omega_i'] = [\omega_i'] \ .$$