

# 3. COROLLARIES OF THE THEOREM OF THE KERNEL

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- (iii)  $(h_{m,1})^{-1}(T) \cap C_m(S)$  is infinite,
- (iv) There exists a finite covering  $S_{m,n}$  of  $S$  such that the fiber product of  $h_{m,n}$  with  $S_{m,n}$  is Galois, Abelian and of positive degree.

Let  $J$  denote the Jacobian scheme of  $C$  over  $S$ . Let  $a: C \rightarrow J$  be an Albanese morphism. Let  $p$  be a prime. Let  $\bar{T}$  denote the closure of  $a(T)$  in  $J(S) \otimes \mathbf{Z}_p$ . Since  $a(T)$  is infinite it follow from the Mordell-Weil Theorem that there exists a  $t \in \bar{T} - a(T)$ . Let  $t_n \in T$  such that  $t - a(t_n) \in p^n J(S)$ . Let  $C_n$  denote the normalization of the fiber-product of  $C$  and  $J$  via the map  $H_n: x \rightarrow p^n x + t_n$  and  $h_{n,1}$  the natural map from  $C_n$  to  $C$ . It follows that  $C_n$  is defined over  $S$  and since  $H_m(J(S)) \supseteq \{t_n: m \mid n\}$  that  $h_{n,1}(C_m(S))$  contains an infinite subset of  $T$ .

All that remains is to exhibit the maps  $h_{m,n}$ . Clearly,  $t_m - t_n = p^n r_{m,n}$  for some  $r_{m,n} \in J(S)$ . Let  $H_{m,n}$  denote the map  $x: p^{m-n}x + r_{m,n}$ . Then  $H_{m,k} = H_{n,k} \circ H_{m,n}$ . It follows that  $H_{m,n}$  pulls back to a morphism  $h_{m,n}: C_m \rightarrow C_n$ . It is easy to see that this morphism becomes Abelian after adjoining the  $p^{m-n}$ -torsion points on  $J$ . This proves the proposition.  $\square$

*Remark.* One can also prove the above proposition with the condition  $n \leq m$  replaced by  $n \mid m$ .

### 3. COROLLARIES OF THE THEOREM OF THE KERNEL

LEMMA 3.3.1. *Suppose  $g: X' \rightarrow X$  is a morphism of smooth proper schemes with geometrically connected fibers over  $S$ . Then if  $\mu \in PF(X'/S)$  and  $s, t \in X(S)$ ,  $(g^*\mu)(s, t) = \mu(g \circ s, g \circ t)$ .*

*Proof.* This follows easily from Lemma 1.3.2.  $\square$

Suppose  $J$  is the Jacobian of  $C$  over  $S$  and  $g$  is an Albanese morphism, then since  $g^*: H_{DR}^1(J/S) \rightarrow H_{DR}^1(C/S)$  is an isomorphism  $g^*: PF(J/S) \rightarrow PF(C/S)$  is an isomorphism.

LEMMA 3.3.2. *Let  $\mu$  be a fixed Picard-Fuchs differential equation on  $C/S$ . Then  $\{\mu(s, t): s, t \in C(S)\}$  lies in a finite dimensional subspace of  $K[S]$  over  $K$ .*

*Proof.* Suppose  $\tilde{\mu} \in PF(J/S)$  such that  $g^*\tilde{\mu} = \mu$ . The lemma follows from the Mordell-Weil theorem which together with the Theorem of the kernel implies that  $J(S)$  modulo the kernel of the homomorphism  $s \rightarrow \tilde{\mu}(e, s)$  is a finitely generated Abelian group.  $\square$

LEMMA 3.3.3. *Suppose  $A$  is an Abelian scheme over  $S$  such that  $[W_{A/S}] = H_{DR}^1(A/S)$  and  $g: C \rightarrow A$  is a non-constant morphism over  $S$ . Fix  $s \in C(S)$ . Then the set  $T = \{t \in C(S) : (g^*\mu)(s, t) = 0 \text{ for all } \mu \in PF(A/S)\}$  is of bounded height.*

*Proof.* Let  $A'$  denote the smallest Abelian subscheme of  $A$  over  $S$  containing  $g(C)$ . Since the map  $g^*: PF(A/S) \rightarrow PF(A'/S)$  is surjective and  $[W_{A/S}] = H_{DR}^1(A/S)$ , it follows from Proposition 2.1.2 that  $g(T)$  is contained in a translation of the group of constant sections of  $A'/S$ . Hence,  $g(T)$  is a set of bounded height. Finally, since  $C \rightarrow g(C)$  is a finite morphism, it follows that  $T$  is a set of bounded height.  $\square$

In particular,

COROLLARY 3.3.4. *Suppose  $A$  is an Abelian scheme over  $S$  such that  $\kappa_{A/S}$  is an isomorphism and  $g: C \rightarrow A$  is a non-constant morphism over  $S$ . Fix  $s \in C(S)$ . Then the set  $\{t \in C(S) : (g^*\mu_\omega)(s, t) = 0 \text{ for all } \omega \in \omega_{A/S}\}$  is of bounded height.*

#### 4. PROOF OF MORDELL'S CONJECTURE

PROPOSITION 3.4.1. *Suppose the kernel of the  $\kappa_{C/S}$  has rank at least 2 over  $K[S]$ , then the points of  $C(S)$  have bounded height.*

*Proof.* Suppose  $C(S)$  contains points of arbitrarily large height. Fix  $s \in C(S)$ . By shrinking  $S$ , if necessary, we may suppose that there exists a function  $z \in K[S]$  such that  $\Omega_S^1 = K[S]dz$  and there exists a finite covering  $\mathcal{L}$  of  $C$  by affine opens  $U$  and functions  $v_U \in \mathcal{O}_C(U)$  such that  $s \in U(S)$ , and  $\Omega_C^1(U)$  is spanned by  $dz$  and  $dv_U$ . We may also suppose that  $s^*v_U = 0$  by replacing  $v_U$  with  $v_U - (s \circ f)^*v_U$  if necessary. For  $U \in \mathcal{L}$ ,  $u \in \mathcal{O}_C(U)$  we define  $\partial_{U,z}u$  and  $\partial_{U,v}u$  by the equation

$$du = \partial_{U,z}udz + \partial_{U,v}udv_U.$$

Then  $\partial_{U,z}$  is a lifting of  $\partial = : \partial/\partial z$ . We set  $\mu(t) = \mu(s, t)$  for

$$\mu \in PF = : PF(C/S)$$

and  $t \in C(S)$ .

Let  $\omega_1$  and  $\omega_2$  be two independent elements in the kernel of  $\kappa_{C/S}$ . It follows that there exist  $\omega'_1$  and  $\omega'_2 \in \omega_{C/S}$  such that

$$\partial[\omega'_i] = [\omega'_i].$$