VII. Yang-Baxter Models

The FKT model has a number of intriguing features. It calculates a determinant of a generalized Alexander matrix. It is the low temperature limit of a generalized Potts model [57].

Is the FKT model a reformulation of the skein model for the Conway polynomial? There are a number of ways to try to generalize the FKT model to obtain a model of the Homfly polynomial. An answer to this question would shed light on the relationship of the FKT model and the Homfly polynomial. (And consequently on the relationship of the Homfly polynomial and the fundamental group of the link.)

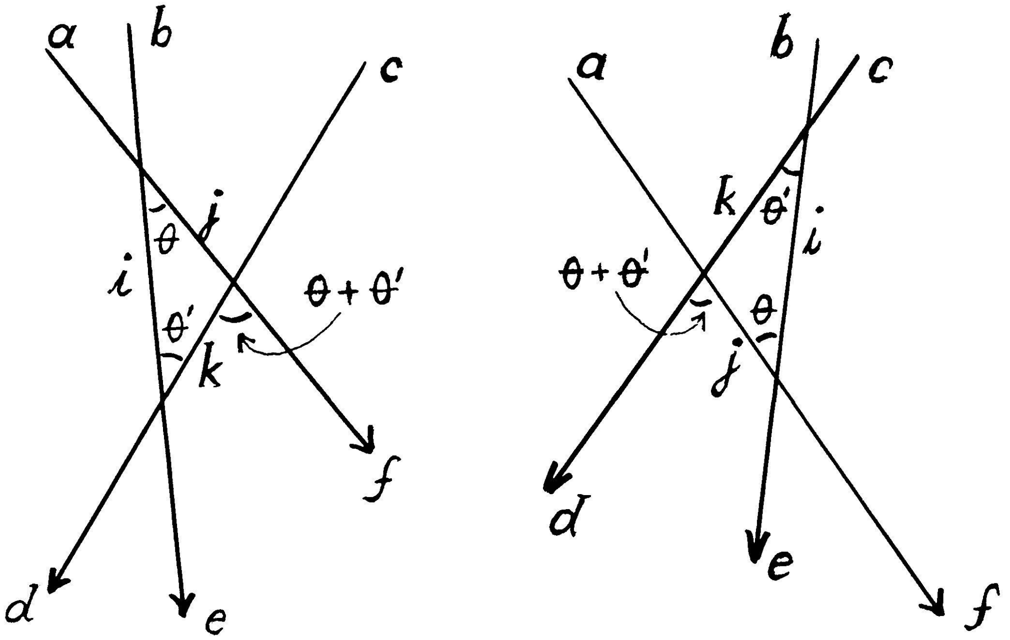
VII. Yang-Baxter Models

I now turn to state models for specializations of the Homfly and Kauffman polynomials that arise from solutions to the Yang-Baxter Equation [10]. These models were devised by Vaughan Jones (Homfly) ([40]) and Volodja Turaev (Kauffman) ([93]). (See also the series of papers ([1], [2], [3], [4], [5], [64]) by Akutsu, Wadati and collaborators.) The reformulation of these models as given here is due to the author (compare [55], [58]).

The Yang-Baxter Equation arises in the study of two-dimensional statistical mechanics models [10] and also in the study of 1 + 1 (1 space dimension, 1 time dimension) quantum field theory ([25], [100]). In the latter case, the motivation and relationship with knot theory is easiest to explain.

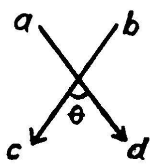
Regard a crossing in a universe (shadow of a link diagram) as a diagram for the interaction of two particles. Label the in-going and out-going lines of an oriented crossing with the "spins" of these particles. (Mathematically, spin is a generic term for a label chosen from an ordered index set $\mathscr{I}$. In applications it may denote the spin of a particle, or it may denote charge or some other intrinsic quantity.) The angle between the crossing segments can be regarded as an indicator of their relative momentum (rapidity). For each assignment of spins and each angle θ there will be a matrix element that, in the physical context, measures the amplitude (complex probability amplitude) for the process with these spins and rapidity.

The S matrix, $S^{ab}_{cd}(\theta)$, is said to be *factorized* if it satisfies the equations shown in Figure 8. This matrix equation is the Yang-Baxter Equation. Physically, it means that amplitudes for multi-particle interactions can be calculated from the two-particle scattering amplitude.



$$\sum_{i,j,k\in\mathcal{I}} S^{ab}_{ij}(\theta)\, S^{jc}_{kf}(\theta+\theta')\, S^{ik}_{de}(\theta')$$
$$= \sum_{i,j,k\in\mathcal{I}} S^{bc}_{ki}(\theta')\, S^{ak}_{dj}(\theta+\theta')\, S^{ji}_{ef}(\theta)$$

Yang-Baxter Equation



$\longleftrightarrow \quad S^{ab}_{cd}(\theta)$

FIGURE 8

As is evident from Figure 8, the Yang-Baxter Equation expresses an invariance related to the type III (triangle) Reidemeister move in knot theory. Figure 9 illustrates the beginning of this correspondence. In this Figure 1 have matched the interaction picture for the S-matrix with a crossing of positive type, and have matched a related matrix, $\bar{S}(\theta)$, with a crossing of negative type. Here we assume that the matrix product

$$S^{ab}_{ij}(\theta)\, \bar{S}^{ij}_{cd}(-\theta) = \delta^a_c\, \delta^b_d$$
(sum on i and j)

is the identity matrix (indicated with Kronecker deltas above). In this form, both crossing and reversed crossing measure the same underlying momentum — corresponding to the (counterclockwise) measure of the angle between the crossing lines.

Switching the crossing corresponds to this step in inverting the S-matrix (that must be combined with a reverse momentum difference to actually obtain the inverse). In the case of a *special S-matrix* (see below and Figure 11) we will accomplish the momentum change with extra interactions (angles in the diagram) so that a crossing and its reverse can cancel.

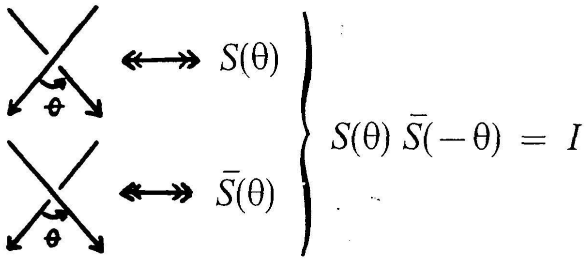
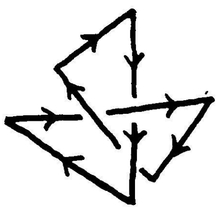


FIGURE 9

It is a curious and deep fact that there is this correspondence between a topological move for weaves in three-dimensional space and the factorizability condition for the S-matrix.

In fact, by assuming that the solution to the Yang-Baxter equation has a special form, one can produce state models for link invariants! In the first version discussed here I shall use piecewise linear (pl) link diagrams. In a pl diagram it will be assumes that each segment of a crossing forms a straight line at the crossing. Along with crossings there are isolated vertices



The angle at a vertex is measured as the angle between the incoming segment and the outgoing segment with direction as shown in Figure 10.

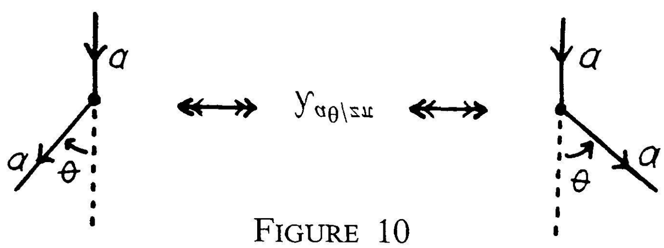


FIGURE 10

One may think of the isolated vertices as interactions with an external field that causes the trajectory of the particle to change direction.

It is assumed that *spin is preserved at all sites of interaction.* Thus, at a 4-vertex we have $a + b = c + d$, and at a 2-vertex we have that the incoming and outgoing spins are identical.

We shall assume that $S(\theta)$ has the following special form:

$$S^{ab}_{cd}(\theta) = R^{ab}_{cd}\, \lambda^{\theta(d-a)/2\pi}$$

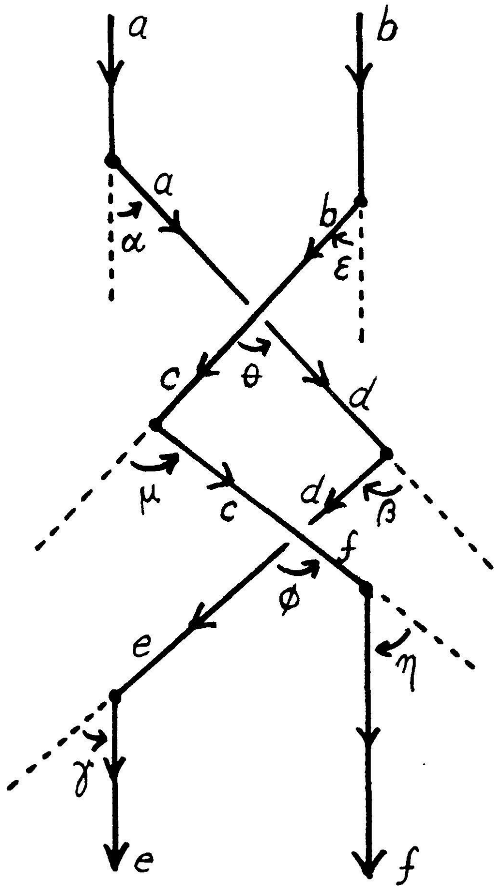
where λ is an (as yet) unspecified variable, and R is an invertible matrix, with no θ-dependence. We also assume that $a + b = c + d$ whenever $R^{ab}_{cd} \neq 0$. Call such an S a *special S-matrix.*

Note that $R = S(0)$ satisfies the Yang-Baxter equation without the angular parameter. Let $\bar{R}$ denote the inverse matrix to R, and let $\bar{S}$ be defined so that $\bar{S}(0) = \bar{R}$:

$$\bar{S}^{ab}_{cd}(\theta) = \bar{R}^{ab}_{cd}\, \lambda^{\theta(d-a)/2\pi}.$$

Remark. A little Euclidean Geometry shows that the oriented type II move works perfectly with respect to this definition when $\bar{S}$ is associated with a crossing of negative type. See Figure 11.

Define the following *vertex weights* for piecewise-linear link diagrams as indicated in Figure 12. In this Figure the local state configuration is shown in the right-hand of the bracket, and the crossing type in the left-hand half.

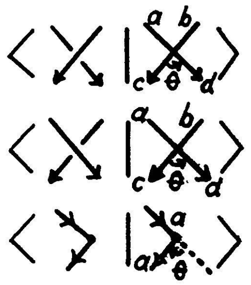


$\alpha + \beta + \gamma = 0$
$\varepsilon + \mu + \eta = 0$
$\theta + \phi = \mu - \beta$

$a + b = c + d$
$c + d = e + f$
(spin conservation)
$\Rightarrow a\alpha + b\varepsilon + (d-a)\theta + \mu c + \beta d + (f-c)\phi + \gamma e + \eta f = 0$
$\Rightarrow$ product of vertex weights for diagram on left is $R^{ab}_{cd}\,\bar{R}^{cd}_{ef}$
$\Rightarrow$ invariance under pl type II move (by $R\bar{R} = I$).

Geometry of Oriented II — Move

FIGURE 11



$$= R^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}$$

$$= \bar{R}^{ab}_{cd}\,\lambda^{\theta(d-a)/2\pi}$$

$$= \lambda^{a\theta/2\pi}$$

Vertex Weights

FIGURE 12

For this Yang-Baxter model, a *state* of the oriented universe U underlying a given link diagram K is *any assignment of spins (from the index set* $\mathcal{J}$*) to the edges of* U (modulo conservation of spin).

Given a diagram K and a state σ, let $<K \mid \sigma>$ denote the *product* of these vertex weights. Let $<K>$ denote the sum of such products taken over all (spin-conserving) states.

LEMMA 7.1. *Let* $S(\theta)$ *be a special factorized S-matrix. Let the state summation*

$$<K> = \sum_{\sigma} <K \mid \sigma>$$

be defined on oriented link diagrams K *as explained above. Then, if* $R = S(0)$ *satisfies the additional condition*

$$(*) \qquad \sum_{i,\,j\in\mathcal{J}} R^{ja}_{ic}\,\bar{R}^{ie}_{jf}\,\lambda^{\frac{c-j}{2}}\,\lambda^{\frac{f-i}{2}} = \delta^{a}_{f}\,\delta^{e}_{c}$$

$\Leftrightarrow$ $S(-\pi)\,\bar{S}(\pi) = I$ where

$$S^{ab}_{cd} = S^{ca}_{db}, \qquad S^{ab}_{cd} = S^{bd}_{ac}$$

$\Leftrightarrow$ Cross Channel Inversion

$\Leftrightarrow$ Inversion under II-move with reverse orientation

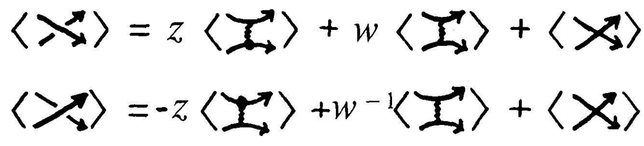
then $<K>$ *is an invariant of regular isotopy.*

(See [40] or [58] for a proof of this lemma.)

Remark. There is a corresponding state model and lemma for unoriented links.

The lemma shows that any angle-free, invertible solution R of the Yang-Baxter Equation gives rise to a regular isotopy invariant of knots and links (by choosing λ to satisfy (*)).

We now exhibit two such solutions in a form that emphasizes their formal similarity to the skein models. In these models the angle (rapidity) terms can all be relegated to counting the number of circuits in a state (with multiplicity). In the Homfly case the model takes the form



$(z = w - w^{-1})$

Note the similarity of the formalism with that for the skein model. Here however, I adopt the convention that *the dotted segment has a smaller spin* than the un-dotted segment. The local state without dots has equal spins on its lines. Spins must be different for crossing lines. The vertex weights of the expansion correspond to a particular solution of the Yang-Baxter Equation.

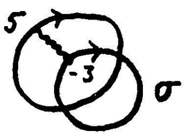
In this model a state σ is a splitting of the universe (i.e. splice a subset of its crossings) and a labelling of the circuits by spins. (The circuits are not allowed to cross themselves.) The value of a state is $\lambda^{\|\sigma\|}$

where $$\|\sigma\| = \sum_{\substack{\text{circuits} \\ C \text{ in } \sigma}} \text{label}(C) \cdot \text{rot}(C)$$

with

$$\text{rot}(\circlearrowleft) = +1, \quad \text{rot}(\circlearrowright) = -1.$$

e.g.



$$\sigma \Rightarrow \|\sigma\| = 5(-1) - 3(-1) = -2.$$

and label(C) is the spin assigned to the (edges of) the circuit C. In this model the state value $\lambda^{\|\sigma\|}$ is a summary of all the angle contributions in the pl formulation. The weights from the set $W = \{z, -z, w, w^{-1}, 0, 1\}$ are the values taken by the angle-independent part R of the S-matrix. This model is expressed for arbitrary link diagrams in the form

$$<K> = \sum_{\sigma} <K \mid \sigma> \lambda^{\|\sigma\|}$$

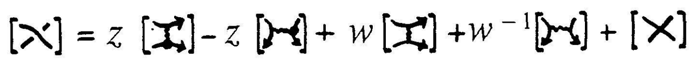
where $<K \mid \sigma>$ denotes the product of vertex weights from the set W arising from the expansion given above.

A particularly nice model occurs for index set in the form

$$\{-n, -n+2, \ldots, n-2, n\} \qquad \text{with} \qquad \lambda = w\,.$$

This gives a series of one-variable specializations of the Homfly polynomial. (See [40], [58], [93].) *Is there a Yang-Baxter model for the full Homfly polynomial?* This is an open question.

A similar approach works for the Dubrovnik form of the Kauffman polynomial. See [58], [93]. The expansion formula has the appearance.



(It is understood that reversing the orientation of a line is accompanied by the negation of its spin.) Once again, the dot on a line means that it has smaller spin.

VIII. Applications and questions

This section is devoted to a few applications of the skein and state models and related questions.

1. Let ∇_K denote the Conway polynomial. The skein model is embodied in the formula of section 6:

$$\nabla_K = \sum_{L, |L| = 1} (-1)^{t-(L)} z^{t(L)}$$

from which we see easily that

$$\max \deg \nabla_K \leqslant V - S + 1 = \rho(K)$$

where V is the number of crossings in the diagram K, S is the number of Seifert circuits (the set of circuits obtained by splicing all crossings of K). On knows that $\rho(K) = \operatorname{rank}(H_1(F))$ where F is the Seifert spanning surface [42] corresponding to the diagram K. If K is an alternating link then $\max \deg \nabla_K = \rho(K)$ [76]. This is generalized to the class of alternative links in [42], using the FKT model. Is there a proof using the skein model? [1])

In the case where all the crossings are of positive type, we see from the skein model that all terms of ∇_K are positive, and it is then easy to see that the highest degree term is of degree $\rho(K)$.

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[1]) *Note added in proof:* A proof using the skein model for the theorem on alternative links has been found by John Mathias — University of Maryland, May 1989.