# 5. Invariants in the case that f has only transversal $A_1$ singularities

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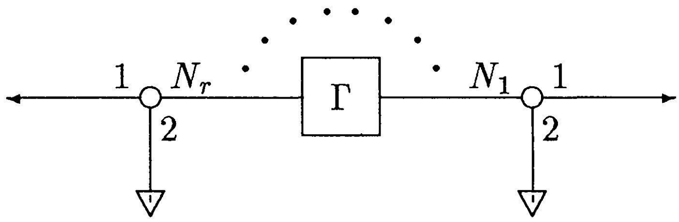
*Proof.* If $q = 1$ the theorem is clear. Now suppose $q > 1$. Consider the behaviour of the polynomials $\Delta_0, \Delta^1$ and $\Delta'$ under this splice operation. Splicing introduces a new edge $E$ which contributes to $\Delta^1$ with a factor $t^q - 1$. This introduces new $2 \times 2$-Jordan blocks. Both splice components have $\sum_{i=1}^{q-1} \left( -\frac{i}{q} \right)$ in their spectrum (coming from $\Delta'$). But, as both eigenvalues in a $2 \times 2$-block are of different weight, $L$ has $\sum_{i=1}^{q-1} \left( -\frac{i}{q} \right) + \left( \frac{i}{q} \right)$ instead of the sum of both parts. It is clear from theorem 4.8 that all other parts of the spectra of $L'$ and $L''$ have to be added. □

## 5. Invariants in the case that $f$ has only transversal $A_1$ singularities

In this section we describe the topology and equation of a topological series that belongs to a non-isolated singularity with only transversal $A_1$ singularities.

Throughout this section, $f \in \mathscr{O}$ is of the form $f = f_1^2 \cdots f_r^2 g$, with $f_1, \ldots, f_r$ irreducible and $g$ reduced. The critical set of $f$ is $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$, and the transverse type of $f$ along $\Sigma_i$ is $A_1$. For all $i \in \{1, \ldots, r\}$, we have numbers $N_{0i}$ and $c_i$ as defined in section 3.3. Let $N_i > N_{0i}$ $(1 \leqslant i \leqslant r)$. According to theorem 3.4, a typical element of the series belonging to $f$ has the topological type (EN-diagram) $\Gamma^*$:

$\Gamma^* =$




That is: each arrow of the EN-diagram $\Gamma$ of $f$ belonging to a double component, is replaced in the way described in theorem 3.4. So varying the $N_i$ will give us the complete series belonging to $f$.

The following two propositions are easy consequences of theorem 3.4. Let $N = (N_1, \ldots, N_r)$ and let $f_N$ have topological type $\Gamma^*$.

5.1. Proposition. *Let $\Delta_*[f]$ and $\Delta_*[f_N]$ be the $\Delta_*$ of $f$ and $f_N$ respectively. Then:*

$$\Delta_*[f_N](t) = \Delta_*[f](t) \cdot \prod_{i=1}^{r} \left( t^{N_i + c_i} - (-1)^{N_i} \right). \qquad \square$$

5.2. PROPOSITION. *Let* $\mu_\infty$ *be the Milnor number of* $f$ *and* $\mu_N$ *that of* $f_N$. *Let* $\mu_0 \in \{1, 2\}$ *be the number of connected components of the Milnor fibre of* $f$. *Then:*

$$\mu_N = \mu_\infty - \mu_0 + 1 + \sum_{i=1}^{r} (N_i + c_i) .$$

*The numbers* $\mu_\infty, \mu_0$ *and* $c_i$ $(1 \leqslant i \leqslant r)$ *depend only on* $f$. □

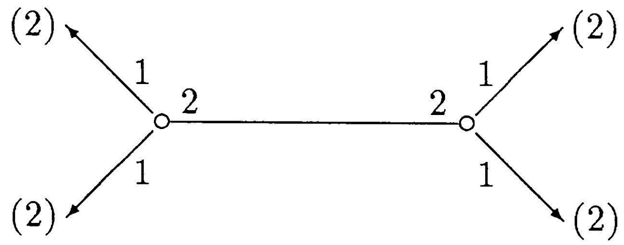
5.3. We conclude this list of topological invariants with the formula of the spectrum of a series (see section 4).

PROPOSITION. *Define for* $1 \leqslant i \leqslant r$: $\gamma_i = 0$ *if* $c_i$ *is even and* $\gamma_i = 1/2$ *when* $c_i$ *is odd. Write* $\nu_i = N_i + c_i$. *Then:*

$$\mathrm{Sp}(f_N) = \mathrm{Sp}(f) + \sum_{i=1}^{r} \sum_{j=0}^{\nu_i - 1} \left(\frac{1}{2} - \frac{\gamma_i + j}{\nu_i}\right)$$

*Proof.* One can use the proof of [St], theorem 4.5, but it is also possible to work out the various cases using the method of section 4. For the proof of [St], the following observation is needed. Let $F_i'$ be a transversal slice transverse to $\Sigma_i$ (in this case $F_i'$ consists of two points — the transverse type is $A_1$). Let $T_i$: $H_0(F_i') \to H_0(F_i')$ be the monodromy of the local system over the punctured disc $\Sigma_i \backslash \{0\}$. Then it is well-known that $T_i$ is the identity if $c_i$ is even and $-$ identity if $c_i$ is odd. In fact, even if the transversal type is not $A_1$, the following holds. Let $t_i$: $H_0(F_i') \to H_0(F_i')$ be the Milnor fibration monodromy of $f$ restricted to a transversal slice through $x \in \Sigma_i$. Then $t_i$ is a cyclic permutation of the finite number of points in $F_i'$, and $T_i = t_i^{-c_i}$. □

*Example.* Let $f(x, y) = (y^2 - x^4)^2 (x^2 - y^4)^2$; its EN-diagram is:



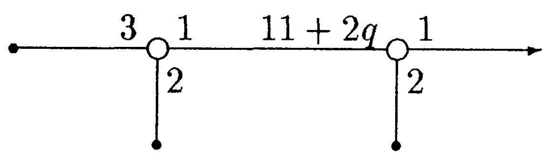

Observe that according to the proposition, the spectrum of $f_N$ is independent of the order of $N_1, \ldots, N_4$. If we take $N = (5, 5, 6, 6)$ and $N = (5, 6, 5, 6)$ we get the same spectrum but different topological types, because the ENdiagrams are not equivalent. This is the counterexample to the spectrum conjecture found by Steenbrink and Stevens, cf. [SSS].

## 6. Equations

In this section we discuss the equations of series: what do we have to add to $f$ to obtain a required element of its series?

In the example $W^{\#}$ at the beginning of section 4, we had:

$W^{\#}_{1,2q-1} : (y^2 - x^3)^2 + x^{4+q}y$




The Puiseux expansion of $W^{\#}_{1,\infty}$: $f(x, y) = (y^2 - x^3)^2$, is $x = t^2, y = t^3$. When we substitute this in $x^{4+q}y$, we get $t^{11+2q}$, which is just the number $N$ in the EN-diagram.

More generally, it appears that adding $\varphi \in \mathscr{O}$ with $\varphi(t^2, t^3)$ of order $11 + 2q$, gives the same result, although there are various kinds of exceptions.

In theorem 6.5 below, we give conditions on $\varphi$ such that $f + \varepsilon\varphi$ has the required type, where $\varepsilon$ is introduced in order to fulfil transversality properties. This avoids exceptional cases such as when $f(x, y) = y^2$ and $\varphi(x, y) = 2x^k y + x^{2k}$, the sum is then a non-isolated singularity.

Again, $f$ has only transversal $A_1$ singularities; but the following lemma is valid in greater generality.

6.1. Lemma. *Let $f, \psi \in \mathscr{O}$ and assume $f$ has a non-isolated singularity. If for all small $\varepsilon > 0$, $f + \varepsilon\psi$ has a singularity topologically equivalent to $f$, then for almost all $\varepsilon$ the zero sets of $f$ and $f + \varepsilon\psi$ are equal.*

*Proof.* First take $f(x, y) = y^n$ with $n > 1$. Assume that for no $\varepsilon$, $f$ and $f + \varepsilon\psi$ have the same zero set. Then we may assume $f + \varepsilon\psi = \big(y + F(x, \varepsilon)\big)^n$ where $F(x, \varepsilon) \not\equiv 0$, regarded also as a function, of $\varepsilon$, can be written as

$$F(x, \varepsilon) = \sum_{i>0} a_i(\varepsilon) x^i .$$

Here $a_i(\varepsilon)$ may have positive fractional powers of $\varepsilon$. $f + \varepsilon\psi$ is linear in $\varepsilon$. By writing out the equation