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Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **36 (1990)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

Persistenter Link: https://doi.org/10.5169/seals-57906

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L'Enseignement Mathématique, t. 36 (1990), p. 143-174

VALUES OF QUADRATIC FORMS AT INTEGRAL POINTS: AN ELEMENTARY APPROACH

by S. G. DANI and G. A. MARGULIS

In response to a longstanding conjecture due to Oppenheim, G. A. Margulis proved (cf. [17] and [20]) that if Q is a nondegenerate indefinite quadratic form on $\mathbf{R}^n, n \geqslant 3$, which is not a multiple of a rational form then for any $\varepsilon > 0$ there exists $p \in \mathbf{Z}^n - \{0\}$ such that $0 < |Q(p)| < \varepsilon$; this also implies, by a well-known number-theoretic method (cf. [14], §5) that for any $a \in \mathbf{R}$ and $\varepsilon > 0$ there exists $p \in \mathbf{Z}^n$ such that

(i) $$0 < |Q(p) - a| < \varepsilon .$$

Subsequently it was proved in [7] (see also [6]) that the element p as in (i) can also be chosen to be primitive (namely such that the g.c.d. of the coordinates is 1). Further, we also proved that if Q is a quadratic form as above and B is the corresponding bilinear form (defined by $B(x, y) = \{Q(x+y) - Q(x-y)\}/4$ for all $x, y \in \mathbf{R}^n$) and $a, b, c \in \mathbf{R}$ are such that there exist $x, y \in \mathbf{R}^n$ for which $Q(x) = a, Q(y) = b$ and $B(x, y) = c$ then for any $\varepsilon > 0$ there exist primitive integral points p and q such that

(ii) $$|Q(p) - a| < \varepsilon, |Q(q) - b| < \varepsilon \quad \text{and} \quad |B(p, q) - c| < \varepsilon .$$

The method of proof in both [20] and [7] is based on studying the orbits on $SL(3, \mathbf{R})/SL(3, \mathbf{Z})$ of the action, on the left, of certain subgroups of $SL(3, \mathbf{R})$. In [7] it was proved that if H is the subgroup of $SL(3, \mathbf{R})$ consisting of all elements leaving invariant a given nondegenerate indefinite quadratic form on $\mathbf{R}^3$ then every orbit of H is either closed or dense; this enables one to deduce the assertion about the existence of primitive integral solutions to (i) as also (ii) under the conditions as above. The earlier proof of the Oppenheim conjecture in [20] is based on showing the relatively compact H-orbits to be closed and certain other supplementary observations (cf. [20]).

The argument in [7], in its entirety, involves various deep theorems on Lie groups, algebraic groups, ergodic theory and unitary representations. Interestingly it turns out that if one is to look only for the existence of primitive integral solutions to (i) then, with some modifications, the argument in [7] can be arranged to yield a proof which not only does not involve any deep theorems but does not involve even any familiarity with the topics mentioned above. The proof is accessible to anyone having gone through basic courses in linear algebra and topological groups! Needless to say that in view of the general nature of the result and the fact that it already implies the Oppenheim conjecture, it is worthwhile to record such a proof. That is the purpose of the present article. We have tried to arrange it so that while a novice should have as little difficulty as possible in understanding the proof, an expert should be able to run through the key ideas, getting a quick understanding of the proof. Many details are included to make the presentation complete.

One major aspect of the present simplification is an observation that to prove the existence of primitive integral solutions to (i) (cf. Main Theorem below) it is enough to prove that all the H-orbits (H as above) which are not closed contain orbits of certain one-parameter semigroups not contained in H (cf. Proposition 8); that is, one does not need the full strength of the assertion in [7] that all such orbits are dense in $SL(3,\mathbf{R})/SL(3,\mathbf{Z})$. Thus, the Main theorem here can be deduced from Proposition 4.1 of [7] rather than Theorem 2 of [7]. The observation is supplemented by some further simplifications to make the proof accessible by elementary methods.

We conclude this introduction with the following acknowledgement and then go on to a formal statement of the Main Theorem.

Acknowledgement. A preliminary version of this was prepared for presentation at the Swiss Seminar in Bern, by the first named author. The author would like to thank the "Troisième cycle romand de mathématiques", Switzerland, for support. He would also like to thank Professor A. Haefliger for encouragement, in the context of the paper.

Let $\mathbf{R}^n, n \geqslant 2$, be the n-dimensional vector space over $\mathbf{R}$, viewed as the space of all n-rowed column matrices with entries in $\mathbf{R}$, equipped with the usual topology. Let $e_1, \cdots, e_n$ be the standard basis of $\mathbf{R}^n$; e_i is the column matrix with 1 in ith row and 0 in all other rows. A $p \in \mathbf{R}^n$ is said to be an integral point if all its entries are integers; namely if it is of the form $p = p_1 e_1 + \cdots + p_n e_n$ where $p_1, \cdots, p_n \in \mathbf{Z}$. We denote by $\mathbf{Z}^n$ the set of all integral points in $\mathbf{R}^n$. An integral point $p = p_1 e_1 + \cdots + p_n e_n$ is said to be primitive if the g.c.d. of $p_1, \cdots, p_n$ is 1 or equivalently if $k^{-1}p$ is not an integral point for any integer $k \geqslant 2$. We denote by $\mathfrak{p}(\mathbf{Z}^n)$ the set of all primitive integral points in $\mathbf{Z}^n$.

We recall that a quadratic form on $\mathbf{R}^n$ is a function of the form

$$Q\left(\sum_{i=1}^{n} p_i e_i\right) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} p_i p_j \quad \text{for all} \quad p_1, \cdots, p_n \in \mathbf{R}\,,$$

where (a_{ij}) is a symmetric $n \times n$ matrix with real entries; (a_{ij}) is called the matrix of Q. A quadratic form is nondegenerate if and only if the matrix is nonsingular. A quadratic form Q is said to be indefinite if there exists $p = p_1 e_1 + \cdots + p_n e_n \in \mathbf{R}^n$ with $p_i \neq 0$ for some $i = 1, \cdots, n$, such that $Q(p) = 0$. Also, a quadratic form is said to be rational if its matrix is rational (that is, has rational entries).

MAIN THEOREM. *Let Q be a nondegenerate indefinite quadratic form on $\mathbf{R}^n, n \geqslant 3$. Suppose that cQ is not a rational quadratic form for any $c > 0$. Then $Q(\mathrm{p}(\mathbf{Z}^n))$ is a dense subset of $\mathbf{R}$; in other words, for any $a \in \mathbf{R}$ and $\varepsilon > 0$ there exists a primitive integral point p such that*

$$|Q(p) - a| < \varepsilon\,.$$

We begin the proof with some general results. The following simple observation was first noted in [17] and played a crucial role in the original proof of the Oppenheim conjecture.

1. LEMMA. *Let G be a Hausdorff topological group and let S be a Hausdorff topological space with a given continuous G-action on it. Let A and B be two closed subgroups of G and let X and Y be closed subsets of S, invariant under the actions of A and B respectively. Suppose further that Y is compact. Let M be a subset of G such that $mY \cap X$ is nonempty for all $m \in M$. Then $gY \cap X$ is nonempty for all $g \in \overline{AMB}$. Further, if C is a subgroup of $A \cap B$ such that Cy is dense in Y for all $y \in Y$ then $gY \subset X$ for all $g \in \overline{AMB} \cap N(C)$, where $N(C)$ denotes the normaliser of C in G.*

Proof. If $g = amb$, where $a \in A, m \in M$ and $b \in B$ then $gY \cap X = a(mbY \cap a^{-1}X) = a(mY \cap X)$ and hence it is nonempty. Thus the set $T := \{g \in G | gY \cap X \neq \emptyset\}$ contains AMB. On the other hand since Y is compact and X is closed, a direct argument shows that T is closed. Hence T contains $\overline{AMB}$, which is precisely the first assertion in the Lemma. Now let C be a closed subgroup as in the hypothesis and let $g \in \overline{AMB} \cap N(C)$. Since $g \in \overline{AMB}$, by the first part there exists $y \in Y$ such that $gy \in X$. Since X is a closed A-invariant subset and $C \subset A$ this yields that $\overline{Cgy} \subset X$. On the other hand since $g \in N(C)$, $Cgy = gCy$ and by the condition on C the latter is dense in gY. Therefore $gY \subset X$. This proves the Lemma.

To be able to apply the Lemma fruitfully one needs to know, in the appropriate context, "enough" new elements in the set $\overline{AMB}$ as above. In our context this is ensured by a simple property of 'unipotent one-parameter groups of linear transformations' which we now recall.

Let E be a finite-dimensional vector space over $\mathbf{R}$ and let $\mathscr{L}(E)$ denote the space of all linear transformations of E into itself. We consider both E and $\mathscr{L}(E)$ equipped with their usual topologies. For any $\tau \in \mathscr{L}(E)$ the sequence $\{\sum_{i=0}^{j} \tau^i/i!\}$ (τ^0 is the identity transformation by convention) converges as $j \to \infty$, to an element of $\mathscr{L}(E)$, denoted by $\exp \tau$. The map $\tau \mapsto \exp \tau$, of $\mathscr{L}(E)$ into itself, is continuous. A linear transformation τ is said to be nilpotent if there exists a natural number k such that $\tau^k = 0$, the zero transformation in $\mathscr{L}(E)$. A family $\{u(t)\}_{t\in\mathbf{R}}$ in $\mathscr{L}(E)$ is called a *unipotent one − parameter group of linear transformations* if there exists a nilpotent linear transformation τ of E such that $u(t) = \exp t\tau$ for all $t \in \mathbf{R}$. A map $f: \mathbf{R} \to E$ is said to be a polynomial map if there exists a basis $e_1, \cdots, e_d$ (where d = dimension of E) and real polynomials $f_1, \cdots, f_d$ such that $f(t) = f_1(t)e_1 + \cdots + f_d(t)e_d$ for all $t \in \mathbf{R}$; if such a basis exists then the components of $f(t)$ with respect to any basis are polynomials in t. We note that if $\{\exp t\tau\}$ is a unipotent one-parameter group of linear transformations of E and $p \in E$ then $t \mapsto (\exp t\tau)(p)$ is a polynomial map. For the proof of the main theorem we need the following lemma; it is a slightly weaker version of Lemma 2.2 of [7] and is related to Lemma 1 of [1] and Lemma 13 of [20].

2. LEMMA. *Let $\{u(t)\}$ be a unipotent one-parameter group of linear transformations of a finite-dimensional* $\mathbf{R}$*-vector space E. Let F denote the vector subspace of E defined by*

$$F = \{p \in E \mid u(t)(p) = p \quad \text{for all} \quad t \in \mathbf{R}\}.$$

Let M_0 be a subset of $E - F$ and let $p_0 \in \overline{M_0} \cap F$. Then there exist a nonconstant polynomial map $\varphi: \mathbf{R} \to F$ and sequences $\{m_i\}$ in M_0 and $\{t_i\}$ in $\mathbf{R}$ *such that $\varphi(0) = p_0$ and for any $s \in \mathbf{R}$, $u(st_i)(m_i) \to \varphi(s)$ as $i \to \infty$.*

Proof. Let τ be a nilpotent linear transformation of E such that $u(t) = \exp t\tau$ for all $t \in \mathbf{R}$. By the Jordan canonical form (cf. [11], [21] or [25], for instance) there exists a basis $\{e_j^{(k)}\}$ where the indices vary over a set of the form $\{(j, k) \mid 1 \leqslant j \leqslant r_k \text{ and } 1 \leqslant k \leqslant l\}$, l and $r_1, \cdots, r_l$ being fixed natural numbers, such that for all $k = 1, \cdots, l$

$$\tau(e_1^{(k)}) = 0 \quad \text{and} \quad \tau(e_j^{(k)}) = e_{j-1}^{(k)} \quad \text{for all} \quad 2 \leqslant j \leqslant r_k .$$

A straightforward computation then shows that

$$u(t)\,(e_j^{(k)}) = e_j^{(k)} + te_{j-1}^{(k)} + \frac{1}{2}\,t^2 e_{j-2}^{(k)} + \cdots + \frac{1}{(j-1)!}\,t^{j-1} e_1^{(k)}$$

for all j, k as above. In particular this means that F is the subspace spanned by $\{e_1^{(k)} | 1 \leqslant k \leqslant l\}$. For $m \in M_0$ let $m(j, k)$ denote the $e_j^{(k)}$-component of m with respect to the basis $\{e_j^{(k)}\}$ and let

$$\theta(m) = \min\{|m(j, k)|^{-1/(j-1)} | 1 \leqslant k \leqslant l \quad \text{and} \quad 2 \leqslant j \leqslant r_k\} .$$

Then $|m(j, k)\theta^{j-1}(m)| \leqslant 1$ whenever $j \geqslant 2$. Now let $\{m_i\}$ be a sequence in M_0 converging to p_0. By passing to a subsequence and modifying notation, we can arrange so that there exists a pair (j_0, k_0) such that $j_0 \geqslant 2$ and $|m_i(j_0, k_0)\theta^{j_0-1}(m_i)| = 1$ for all i. Passing to a subsequence one again, we may further assume that for each pair (j, k), $1 \leqslant j \leqslant r_k$, $1 \leqslant k \leqslant l$, the sequence $\{m_i(j, k)\theta^{j-1}(m_i)\}$ converges as $i \to \infty$; let $\lambda(j, k)$ denote the limit of the sequence. Observe that $|\lambda(j_0, k_0)| = 1$. Now choose

$$\varphi(s) = \sum_{k=1}^{l} \left(\sum_{j=1}^{r_k} \frac{1}{(j-1)!}\,\lambda(j, k) s^{j-1} \right) e_1^{(k)} \quad \text{for all} \quad s \in \mathbf{R} .$$

Then φ defines a polynomial map of $\mathbf{R}$ into F. Since $|\lambda(j_0, k_0)| = 1$ and $j_0 \geqslant 2$, φ is a nonconstant map. It is straightforward to verify that if $\{m_i\}$ is the sequence as above (after successive reductions) and $t_i = \theta(m_i)$ then for any $s \in R$, $u(st_i)\,(m_i) \to \varphi(s)$ as $i \to \infty$. Also clearly

$$\varphi(0) = \sum_{k=1}^{l} \lambda(1, k) e_1^{(k)} = \lim_{i\to\infty} \sum_{k=1}^{l} m_i(1, k) e_1^{(k)} = p_0 ,$$

since $m_i \to p_0$ and $p_0 \in F$.

We now introduce some notation to be followed throughout. Let $G = SL(3, \mathbf{R})$ be the group of 3×3 matrices with real entries and determinant 1, equipped with the usual topology of componentwise convergence of the entries. For any $t \in \mathbf{R}$ let

$$v_1(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad v_2(t) = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and let

$$V_1 = \{v_1(t)\,|\,t \in \mathbf{R}\}\,,\quad V_2 = \{v_2(t)\,|\,t \in \mathbf{R}\}$$
$$V_2^+ = \{v_2(t)\,|\,t \geqslant 0\} \quad \text{and} \quad V_2^- = \{v_2(t)\,|\,t \leqslant 0\}\,.$$

Also for any $t \in \mathbf{R}^*$ (namely a nonzero real number) let $d(t)$ denote the diagonal matrix $\operatorname{diag}(t, 1, t^{-1})$ and let

$$D = \{d(t)\,|\,t > 0\}\,.$$

As stated before we view $\mathbf{R}^3$ as the space of 3-rowed column-matrices and denote by e_1, e_2, e_3 the standard basis elements. For any (not necessarily square) matrix ξ we denote by ${}^t\xi$ the transpose of ξ. The 3×3 identity matrix will be denoted by I.

Let Q_0 and Q_1 be the quadratic forms on $\mathbf{R}^3$ defined by

$$Q_0(p_1 e_1 + p_2 e_2 + p_3 e_3) = 2p_1 p_3 - p_2^2 \quad \text{and}$$
$$Q_1(p_1 e_1 + p_2 e_2 + p_3 e_3) = p_3^2 \quad \text{for all} \quad p_1, p_2, p_3 \in \mathbf{R}\,.$$

We note that for all $p \in \mathbf{R}^3$ and $t \in \mathbf{R}$,

(iii) $$Q_0\big(v_1(t)p\big) = Q_0\big({}^t v_1(t)p\big) = Q_0\big(d(\pm e^t)p\big) = Q_0(p)$$
$$\text{and} \quad Q_0\big(v_2(t)p\big) = Q_0(p) + 2tQ_1(p)\,.$$

Let

$$H = \{g \in G\,|\,Q_0(gp) = Q_0(p) \quad \text{for all} \quad p \in \mathbf{R}\}$$

Then H is a closed subgroup of G containing V_1 and D.

As for a linear transformation, for any square matrix ξ we denote by $\exp \xi$ the limit of the sequence $\{\sum_{i=0}^{j} \xi^i/i!\}$. If ξ is the matrix representing a linear transformation τ with respect to a basis then $\exp \xi$ is the matrix representing $\exp \tau$ with respect to the same basis. Let

(iv) $$\nu_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \nu_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Then we see that $\nu_1^3 = \nu_2^2 = 0$ and that

(v) $$\exp t\nu_1 = v_1(t)\,,\ \exp t\nu_2 = v_2(t) \quad \text{and} \quad \exp t\delta = d(e^t)$$

for all $t \in \mathbf{R}$.

We next apply Lemma 2 and deduce the following result which is one of the main ingredients of the proof of the main theorem. We give two proofs of the assertion. The first proof uses elementary calculus of several variables (strictly speaking, the arguments are motivated by some Lie group theory which however is not involved directly). The second proof is based on certain standard arguments in topological groups.

3. PROPOSITION. *Let* M *be a subset of* $G - HV_2$ *such that* $I \in \overline{M}$. *Then either* V_2^+ *or* V_2^- *is contained in* $\overline{HMV_1}$.

First proof. Let $E = M(3, \mathbf{R})$ be the space of all 3×3 matrices with real entries, equipped with the usual topology. Let P be the subspace of E defined by

$$P = \{\xi = (\xi_{ij}) \mid \xi_{11} - \xi_{33} = \xi_{12} + \xi_{23} = \xi_{21} + \xi_{32} = \xi_{11} + \xi_{22} + \xi_{33} = 0\}.$$

Though we shall not need this fact, it is worth noting that P is the orthocomplement of the Lie subalgebra corresponding to H in the Lie algebra of G, with respect to the Killing form.

We show that given a sequence $\{g_i\}$ in G such that $g_i \to I$, there exist sequences $\{h_i\}$ in H and $\{\eta_i\}$ in P such that $h_i \to I$, $\eta_i \to 0$ and $g_i = h_i(\exp \eta_i)$ for all i. Observe that for $\eta \in P$, the sum of the diagonal entries being zero implies that the sum of the eigenvalues of η is zero and hence the determinant of $\exp \eta$ is 1. Any $\xi \in E$ can be expressed uniquely as $\xi = \alpha I + a\delta + b\nu_1 + c\,{}^t\nu_1 + \eta$ where ν_1 and δ are as in (iv), $\alpha, a, b, c \in \mathbf{R}$ and $\eta \in P$. Consider the map $\psi: E \to E$ defined by $\psi(\alpha I + a\delta + b\nu_1 + c\,{}^t\nu_1 + \eta) = e^{\alpha} d(e^{a}) v_1(b)\,{}^t v_1(c) (\exp \eta)$ for all $\alpha, a, b, c \in \mathbf{R}$ and $\eta \in P$. We note that ψ is a C^1 map, when E is viewed as $\mathbf{R}^9$ with ξ_{ij} as the coordinate variables and that the Jacobian determinant of ψ at the point 0 (namely the zero matrix) is nonzero; in fact the derivative at 0 is the identity map. Hence by the inverse function theorem (cf. [12] for instance) there exists a neighbourhood W of 0 in E such that the restriction of ψ to W is a homeomorphism of W onto a neighbourhood of I in E. Let $\{g_i\}$ be a sequence in G converging to I. Then by the preceding observation there exist sequences $\{\alpha_i\}, \{a_i\}, \{b_i\}$, and $\{c_i\}$ in $\mathbf{R}$ and $\{\eta_i\}$ in P such that each of the sequences converges to zero (in $\mathbf{R}$ or P respectively) and $g_i = e^{\alpha_i} d(e^{a_i}) v_1(b_i)\,{}^t v_1(c_i) (\exp \eta_i)$ for all i. Comparing the determinants we see that $\alpha_i = 0$ for all i. Also in view of (iii), $d(e^{a_i}) v_1(b_i)\,{}^t v_1(c_i) \in H$ for all i. Thus we get the sequences $\{h_i\}$ in H and $\{\eta_i\}$ in P as desired.

Now let M be the subset as in the hypothesis. Then M contains a sequence $\{g_i\}$ such that $g_i \to I$. Let $\{h_i\}$ and $\{\eta_i\}$ be sequences in H and P respectively such that $h_i \to I$, $\eta_i \to 0$ and $g_i = h_i(\exp \eta_i)$ for all i. Let ν_1 be the matrix as in (iv). It is easy to see that for any $\eta \in P$, $\nu_1\eta - \eta\nu_1 \in P$. Let $\tau: P \to P$ be the map defined by $\tau(\eta) = \nu_1\eta - \eta\nu_1$ for all $\eta \in P$. Then τ is a linear transformation of P. Further a straightforward computation using the fact that $\nu_1^3 = 0$ shows that $\tau^5 = 0$, the zero transformation of P. Thus τ is a nilpotent linear transformation. We also note that the corresponding unipotent one-parameter group of linear transformations of P is given by

(vi) $$(\exp t\tau)(\eta) = v_1(t)\eta v_1(-t) \quad \text{for all} \quad t \in \mathbf{R} \quad \text{and} \quad \eta \in P\,.$$

We now apply Lemma 2 to the unipotent one-parameter group $\{\exp t\tau\}$ as above. A direct computation shows that the subspace F of P consisting of all η in P such that $(\exp t\tau)(\eta) = \eta$ for all $t \in \mathbf{R}$ is spanned by the element ν_2 as in (iv). For all i we have $g_i = h_i(\exp \eta_i) \in G - HV_2$ and hence $(\exp \eta_i) \notin V_2$; this implies that $\eta_i \in P - F$ for all i, since F is spanned by ν_2 and $\exp t\nu_2 = v_2(t) \in V_2$ for all $t \in \mathbf{R}$. Applying Lemma 2 with the set $\{\eta_i \mid i = 1, 2, \cdots\}$ and the point 0 in the place of M_0 and p_0 there, we conclude that there exists a nonconstant polynomial map $\varphi\colon \mathbf{R} \to \mathbf{R}$ and sequences $\{i_k\}$ and $\{t_k\}$ in $\mathbf{N}$ and $\mathbf{R}$ respectively such that $\varphi(0) = 0$ and for any $s \in \mathbf{R}$, $(\exp st_k\tau)(\eta_{i_k}) \to \varphi(s)\nu_2$ as $k \to \infty$. Then for any $s \in \mathbf{R}$ we have

(vii) $$\begin{aligned} v_1(st_k)(\exp \eta_{i_k})v_1(-st_k) &= \exp\{v_1(st_k)\eta_{i_k}v_1(-st_k)\} \\ &= \exp\{(\exp st_k\tau)(\eta_{i_k})\} \to \exp \varphi(s)\nu_2 = v_2(\varphi(s))\,. \end{aligned}$$

Since $(v_1(st_k)h_{i_k}^{-1})g_{i_k}v_i(-st_k) = v_1(st_k)(\exp \eta_{i_k})v_1(-st_k)$ and since the sequences $\{v_1(st_k)h_{i_k}^{-1}\}$ and $\{g_{i_k}\}$ are contained in H and M respectively, (vii) shows that for all $s \in \mathbf{R}$, $v_2(\varphi(s)) \in \overline{HMV_1}$. Since φ is a nonconstant real polynomial and $\varphi(0) = 0$, the image of φ contains either all positive numbers or all negative numbers. Thus we get that $\overline{HMV_1}$ contains either V_2^+ or V_2^-. This proves the proposition.

Second proof. Let S be the vector space of all symmetric 3×3 matrices with real entries. Let ν_1 be the matrix as in (iv). We observe that for any $\sigma \in S$, ${}^t\nu_1\sigma + \sigma\nu_1$ is also an element of S and that the map $\tau\colon S \to S$ defined by $\tau(\sigma) = -({}^t\nu_1\sigma + \sigma\nu_1)$ is a nilpotent linear transformation; specifically $\tau^5 = 0$ (the zero transformation). We also note that the corresponding unipotent one-parameter group of linear transformations is given by

(viii) $$(\exp t\tau)(\sigma) = {}^tv_1(-t)\sigma v_1(-t) \quad \text{for all} \quad t \in \mathbf{R} \quad \text{and} \quad \sigma \in S\,.$$

Let $F = \{\sigma \in S \mid (\exp t\tau)(\sigma) = \sigma \text{ for all } t \in \mathbf{R}\}$. A straightforward computation shows that F is spanned by the elements σ_0 and σ_1 defined by

(ix) $$\sigma_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

We note that a matrix in F has determinant 1 if and only if it is of the form $\sigma_0 + t\sigma_1$ for some $t \in \mathbf{R}$. Now let M be the subset as in the hypothesis and let $M_0 = \{{}^t g\sigma_0 g \mid g \in M\}$. Then $M_0 \subset S$. We show that $M_0 \cap F = \emptyset$. If possible let $g \in M$ be such that ${}^t g\sigma_0 g \in F$. Since ${}^t g\sigma_0 g$ has determinant 1, by the above observation, there exists $t \in \mathbf{R}$ such that ${}^t g\sigma_0 g = \sigma_0 + 2t\sigma_1$. The latter can be written as ${}^t v_2(t)\sigma_0 v_2(t)$. Thus we get that ${}^t g\sigma_0 g = {}^t v_2(t)\sigma_0 v_2(t)$. Hence ${}^t h\sigma_0 h = \sigma_0$, where $h = gv_2(-t)$. Thus ${}^t p\,{}^t h\sigma_0 hp = {}^t p\sigma_0 p$ for all $p \in \mathbf{R}^3$ and this means that $Q_0(hp) = Q_0(p)$ for all $p \in \mathbf{R}^3$. Therefore $h \in H$. But then $g = hv_2(t) \in HV_2$, which is a contradiction since $M \subset G - HV_2$. Hence $M_0 \cap F = \emptyset$.

We now apply lemma 2 to the unipotent one-parameter group $\{\exp t\tau\}$ and the set M_0 as above and σ_0 in the place of p_0 and conclude that there exist a nonconstant polynomial map $\psi: \mathbf{R} \to F$ and sequences $\{g_i\}$ in M and $\{t_i\}$ in $\mathbf{R}$ such that $\psi(0) = \sigma_0$ and for any $s \in \mathbf{R}$, $(\exp st_i\tau)({}^t g_i\sigma_0 g_i) \to \psi(s)$ as $i \to \infty$; substituting from (viii) we get that ${}^t v_1(-st_i)({}^t g_i\sigma_0 g_i)v_1(-st_i) \to \psi(s)$ as $i \to \infty$. For each s, each matrix in this sequence has determinant 1 and therefore $\psi(s)$ has determinant 1. Since ψ is a polynomial map into F, in view of the remark about elements of F with determinant 1, this implies that there exists a (unique) polynomial φ on $\mathbf{R}$ such that $\psi(s) = \sigma_0 - 2\varphi(s)\sigma_1$ for all $s \in \mathbf{R}$; since ψ is nonconstant, so is φ and since $\psi(0) = \sigma_0$, $\varphi(0) = 0$. Now, for all $s \in \mathbf{R}$ we have

(x) $$\begin{aligned} {}^t v_1(-st_i)({}^t g_i\sigma_0 g_i)v_1(-st_i) \to \psi(s) &= \sigma_0 - 2\varphi(s)\sigma_1 \\ &= {}^t v_2(-\varphi(s))\sigma_0 v_2(-\varphi(s)) . \end{aligned}$$

Now consider the G-action on S given by $(g, \sigma) \mapsto {}^t g^{-1}\sigma g^{-1}$ for all $g \in G$ and $\sigma \in S$. Let $T \subset S$ be the orbit of σ_0 under the action. Any $\sigma \in T$ is indefinite (namely there exists $p \in \mathbf{R}^3$, $p \neq 0$ such that ${}^t p\sigma p = 0$) and has determinant 1. We show, conversely, that if $\sigma \in S$ is indefinite and has determinant 1 then $\sigma \in T$. Consider such a σ. If δ is a diagonal matrix with diagonal entries ± 1 which is equivalent (cogradient) to σ, the conditions on σ imply that exactly one of the diagonal entries is 1. Since this holds for σ_0 as well we get that $\sigma = \rho\sigma_0{}^t\rho$ for some nonsingular matrix ρ (cf. [11] Ch. V, Theorem 6). Then clearly ρ has determinant ± 1 and hence we can choose $g \in G$, $g = \pm\,{}^t\rho^{-1}$ such that $\sigma = {}^t g^{-1}\sigma_0 g^{-1}$; this shows that $\sigma \in T$. Thus T is precisely the set of all indefinite matrices of determinant 1. This implies in particular that T is a closed subset of S, with respect to the usual topology on S. In particular T is locally compact with respect to the induced topology.

If $g \in G$ is such that ${}^t g^{-1} \sigma_0 g^{-1} = \sigma_0$ then ${}^t p\, {}^t g \sigma_0 g p = {}^t p \sigma_0 p$ for all $p \in \mathbf{R}^3$, which implies that $g \in H$. This yields that H is the isotropy subgroup of σ_0 under the G-action as above. Hence we have a canonical bijection $\theta \colon G/H \to T$ defined by $\theta(gH) = {}^t g^{-1} \sigma_0 g^{-1}$. Since G is second countable and T is locally compact the canonical bijection θ is a homeomorphism (cf. [9], Ch. V, § 1, Theorem 8 or [10], (1.6.1) for instance), when G/H is equipped with the quotient topology.

Observe that in view of (x), for any $s \in \mathbf{R}$, $\theta\big(v_1(st_i) g_i^{-1} H\big) \to \theta\big(v_2(\varphi(s))H\big)$. Since θ is a homeomorphism, this implies that for all $s \in \mathbf{R}$, $v_1(st_i) g_i^{-1} H \to v_2(\varphi(s))H$ in the quotient space G/H. Since $\{g_i\}$ is contained in M this implies, in turn, that $\overline{V_1 M^{-1} H}$ contains $v_2(\varphi(s))$ for all $s \in \mathbf{R}$. Hence $\overline{HMV_1}$, which is the same as $(\overline{V_1 M^{-1} H})^{-1}$, contains $v_2(\varphi(s))$ for all $s \in \mathbf{R}$. Since φ is a nonconstant real polynomial such that $\varphi(0) = 0$, the image of φ contains either all negative numbers or all positive numbers. Hence the preceding conclusion implies that $\overline{HMV_1}$ contains either V_2^+ or V_2^-, thus proving the proposition.

4. Proposition. *Let $h \in H$ and $v \in V_2 - \{I\}$ be such that $vh \in HV_2$. Then h is an upper triangular matrix.*

Proof. Let $h \in H$ and $v = v_2(t)$, $t \neq 0$ be such that $vh \in HV_2$; let $h' \in H$ and $v' = v_2(s)$, $s \in \mathbf{R}$ be such that $vh = h'v'$. By (iii), for any $p \in \mathbf{R}^3$ we have

$$Q_0(vhp) = Q_0(hp) + 2tQ_1(hp) = Q_0(p) + 2tQ_1(hp) \quad \text{and}$$
$$Q_0(h'v'p) = Q_0(v'p) = Q_0(p) + 2sQ_1(p) .$$

Since $vh = h'v'$, this yields that $Q_1(hp) = (s/t)Q_1(p)$ for all $p \in \mathbf{R}^3$. Let L be the plane spanned by e_1 and e_2. Then L is precisely the set on which Q_1 vanishes and hence the preceding conclusion implies that he_1 and he_2 belong to L. Further if $he_1 = p_1 e_1 + p_2 e_2$, where $p_1, p_2 \in \mathbf{R}$, then we have $-p_2^2 = Q_0(p_1 e_1 + p_2 e_2) = Q_0(he_1) = Q_0(e_1) = 0$, which shows that $he_1 = p_1 e_1$. This together with the fact that $he_2 \in L$ implies that h is an upper triangular matrix.

Now let $V = V_1 V_2$. Then V is a closed abelian subgroup of G. Each $d \in D$ normalises the subgroups V_1 and V_2. Therefore DV_1 and DV are subgroups of G. It is straightforward to verify that they are closed subgroups of G. In the sequel we need the following simple property of DV.

5. LEMMA. *Let* Δ *be a discrete subgroup of* DV; *then either* Δ *is contained in* V *or it is a cyclic subgroup generated by an element of the form* vdv^{-1}, *where* $v \in V$ *and* $d \in D - \{I\}$.

Proof. We first note that for any $d \in D - \{I\}$ and $w \in V$ there exists $v \in V$ such that $dw = vdv^{-1}$; such a v may be readily determined, keeping in mind that ve_1, ve_2, ve_3 must be eigenvectors of dw. Now let Δ be a discrete subgroup of DV which is not contained in V. Thus there exist $d \in D - \{I\}$ and $w \in V$ such that $dw \in \Delta$. Let $v \in V$ be such that $dw = vdv^{-1}$. Let $\Delta' = v^{-1}\Delta v$. Then Δ' is a discrete subgroup of DV containing d. Let au, where $a \in D$ and $u \in V$, be any element of Δ. Then $d^j(au)d^{-j} \in \Delta'$ for all j. We see that $d^j ud^{-j} \to I$ either as $j \to \infty$ or as $j \to -\infty$. Hence $d^j(au)d^{-j} = a(d^j ud^{-j}) \to a$ either as $j \to \infty$ or as $j \to -\infty$. Since Δ' is discrete this implies that $a(d^j ud^{-j}) = a$ for some j. Hence $u = I$. This shows that Δ' is contained in D. It is easy to see that any discrete subgroup of D is cyclic. Thus Δ' is a cyclic subgroup of D and, therefore Δ, which is the same as $v\Delta' v^{-1}$, is the cyclic subgroup generated by vdv^{-1}, where $d \in D$ is a generator of Δ'; since Δ is not contained in V, $d \neq I$. This proves the Lemma.

We next note the following simple fact. While an expert may recognise this as an immediate consequence of the fact that H contains a subgroup of index 2 which is Lie-isomorphic to $PSL(2, \mathbf{R})$, it can also be deduced directly as indicated below.

6. PROPOSITION. *H/DV_1 is compact (in the quotient topology).*

Proof. Let $\tilde{C} = \{p \in \mathbf{R}^3 - \{0\} \mid Q_0(p) = 0\}$, viewed as a subspace of $\mathbf{R}^3$, and let C be the quotient space of $\tilde{C}$ under the equivalence relation identifying $p, q \in \tilde{C}$ if there exists $\lambda \in \mathbf{R}$ such that $q = \lambda p$. Then C is a compact space (it is a closed subset of the projective space). For any $p \in \tilde{C}$ let $\bar{p} \in C$ denote the equivalence class of p. Consider the action of H on C defined by $h(\bar{p}) = \overline{hp}$ for all $h \in H$ and $p \in \tilde{C}$; it is easy to see that the action is well defined and continuous. It can be verified directly that for any $\bar{p} \in C$ there exists $h \in H$ such that $h(\overline{e_1}) = \bar{p}$; if $\bar{p} \neq \overline{e_3}$ then we can find h of the form ${}^t v_1(t)$, where $t \in \mathbf{R}$, satisfying this and if $\bar{p} = \overline{e_3}$ we can choose $h = \sigma_0$ as in (ix), which is indeed an element of H. Thus C is the orbit of $\overline{e_1}$. Let R be the isotropy subgroup of $\overline{e_1}$. Since H is second countable and C is compact we get that H/R is homeomorphic to C, and therefore compact, in the quotient topology (cf. [9] Ch. V, §1, or [10], (1.6.1) for instance). It is easy to see that if $h \in H$ then $h \in R$ if and only if either $h \in DV_1$ or $h \in d(-1)DV_1$. Therefore DV_1 is a subgroup of index 2 in R. Since H/R is compact, this implies that H/DV_1 is compact.

Now let $\Gamma = SL(3, \mathbf{Z})$ be the subgroup of G consisting of all matrices with integer entries. We equip G/Γ with the quotient topology and consider the G-action defined by left translation; $g \in G$ acts by taking $h\Gamma$, $h \in G$, to $gh\Gamma$.

7. PROPOSITION. *Let X be a nonempty closed subset of G/Γ. Then the following conditions are satisfied:*

a) *If X is V_1-invariant then it contains a minimal (nonempty) closed V_1-invariant subset and any such subset is compact.*

b) *If X is DV_1-invariant then it contains a minimal (nonempty) closed DV_1-invariant subset and no such subset is a DV_1-orbit.*

Given a nonempty *compact* subset X of G/Γ which is invariant under V_1 or DV_1, a simple application of Zorn's lemma shows that X contains a minimal (nonempty) closed subset invariant under V_1 or DV_1 respectively; we only need to observe that in view of the compactness of X, any family of nonempty closed invariant (under V_1 or DV_1 respectively) subsets, which is totally ordered with respect to the inclusion relation, has a nonempty intersection. Now suppose that Y is a compact subset which is a DV_1-orbit, say $Y = DV_1 y$ where $y \in G/\Gamma$. Let $\Delta = \{g \in DV_1 | gy = y\}$. Then DV_1/Δ is homeomorphic to Y (cf. [9], Ch. V, or [10], (1.6.1)). In particular DV_1/Δ is compact. But Δ is a discrete subgroup of DV_1 (and in turn of DV) and hence by Lemma 5, Δ is either contained in $V_1 (= V \cap DV_1)$ or it is a cyclic subgroup generated by an element of the form udu^{-1} where $d \in D$ and $u \in V_1 (= \{v \in V | vdv^{-1} \in DV_1\})$. In either case we see that DV_1/Δ is noncompact. This is a contradiction showing that there are no compact DV_1-orbits. These observations show that the Proposition holds for compact subsets X.

For a noncompact closed subset the Proposition follows from certain results on the asymptotic behaviour of orbits on G/Γ of unipotent one-parameter groups of matrices. Specifically, we need a 'uniform version' of what is referred to as Margulis' Lemma in [3]. Theorem 1.1 of [7] is a quantitative version of what is needed; in [7] we used it to derive the result as in the present Proposition. The proof of Theorem 1.1 of [7] depends on an elementary (though rather complicated) argument using some properties of polynomials. A weaker (qualitative) version adequate in proving the present Proposition, is somewhat simpler to prove. We are including a proof of a weaker version in this text. However, since it involves considerable digression, we defer it until the Appendix (cf. Theorem A.8). A deduction of the Proposition in the general case is given after the proof of Theorem A.8.

8. PROPOSITION. *Let* $x \in G/\Gamma$ *and let* $X = \overline{Hx}$. *Then either* $X = Hx$ *or there exists* $y \in G/\Gamma$ *such that* $V_2^+ y$ *or* $V_2^- y$ *is contained in* X.

Proof. Since $DV_1 \subset H$, X is DV_1-invariant and therefore by Proposition 7 b) it contains a minimal nonempty closed DV_1-invariant subset, say X_1. By Proposition 7 a) X_1 contains a minimal nonempty closed V_1-invariant subset and any such subset is compact. Let Y be such a subset. We shall show that unless $X = Hx$, $V_2^+ Y$ or $V_2^- Y$ is contained in X. Let $y \in Y$. We divide the proof into three cases as follows.

a) there exists a subset M of $G - HV_2$ such that $I \in \overline{M}$ and $my \in X$ for all $m \in M$.

b) there exists a neighbourhood W of I in G such that $\{g \in W \mid gy \in X\} \subset H$.

c) there exist a neighbourhood Ω of I in G and a sequence $\{v_i\}$ in $V_2 - \{I\}$ such that $\{g \in \Omega \mid gy \in X\} \subset HV_2$, $v_i \to I$ and $v_i y \in X$ for all i.

We first observe that at least one of the three cases must hold. Suppose a) and b) do not hold. Then there exists a neighbourhood Ω of I in G such that $\{g \in \Omega \mid gy \in X\} \subset HV_2$ and there exists a sequence $\{g_i\}$ in $G - H$ such that $g_i \to I$ and $g_i y \in X$ for all i. Without loss of generality we may also assume $\{g_i\}$ to be contained in Ω. By the property of Ω this implies that each g_i can be expressed as $h_i v_i$ where $h_i \in H$ and $v_i \in V_2$. Since $\{g_i\}$ is contained in $G - H$, $v_i \neq I$ for all i. Also for any $p \in \mathbf{R}^3$ we have $Q_0(g_i p) = Q_0(h_i v_i p) = Q_0(v_i p) = Q_0(p) + 2t_i Q_1(p)$ where $\{t_i\}$ is the sequence in $\mathbf{R}$ such that $v_i = v_2(t_i)$ for all i. Since $g_i \to I$, this implies that $t_i Q_1(p) \to 0$ for all $p \in \mathbf{R}^3$. Hence $t_i \to 0$, which means that $v_i = v_2(t_i) \to I$. Also since $g_i y = h_i v_i y \in X$ for all i and X is H-invariant, we get that $v_i y \in X$ for all i. This shows that c) holds.

Case a) In this case we see that X and Y are two closed subsets of G/Γ invariant under H and V_1 respectively, Y is compact and $mY \cap X$ is nonempty (as it contains my) for all $m \in M$. Further since Y is a minimal V_1-invariant closed subset, $V_1 y$ is dense in Y for all $y \in Y$. Under these conditions Lemma 1 implies that $gY \subset X$ for all $g \in \overline{HMV_1} \cap N(V_1)$, $N(V_1)$ being the normaliser of V_1 in G. By Proposition 3, $\overline{HMV_1}$ contains either V_2^+ or V_2^-. Since $V_2 \subset N(V_1)$ we now get that $V_2^+ Y$ or $V_2^- Y$ is contained in X.

Case b) In this case we have $Wy \cap X \subset Hy$. Since Hx is dense in X this implies $Hx \cap Hy$ is nonempty and hence $Hx = Hy$. We next observe that $\overline{X_1 - Hy}$ is a closed DV_1-invariant subset of X_1, not containing y. Since X_1 is a minimal nonempty closed DV_1-invariant set, this implies $\overline{X_1 - Hy}$ is empty. Hence $X_1 \subset Hy$. Since by Proposition 6 H/DV_1 is compact, there exists a compact subset K of H such that $H = K(DV_1)$ (cf. [9], Ch. V for instance). Since X_1 is DV_1-invariant $X_1 = DV_1X_1$ and hence $KX_1 = K(DV_1)X_1 = HX_1$ which shows that the set KX_1 is H-invariant. But since $K \subset H$ and $X_1 \subset Hy, KX_1 \subset Hy$ and hence KX_1 being H-invariant implies that $KX_1 = Hy$. On the other hand since K is compact and X_1 is closed, KX_1 is closed. Thus we get that Hy is closed. As $X = \overline{Hx}$ and $Hx = Hy$ this implies that $X = Hx$, thus settling the case.

Case c) By replacing Ω by a smaller neighbourhood if necessary, we may assume that the following conditions also hold for Ω: i) Ω is open, ii) any $g \in \Omega$ has only positive entries on the diagonal and iii) any element of Ωy can be expressed uniquely as gy, where $g \in \Omega$; the last condition can be ensured since Γ is discrete. We now first deduce that $\Omega y \cap DV_1 y$ is contained in $(\Omega \cap DV)y$. Let $g \in DV_1$ be such that $gy \in \Omega y$; say $gy = wy$ where $w \in \Omega$. Then for all i we have $gv_i y = (gv_i g^{-1})gy = (gv_i g^{-1})wy$. Since Ω is a neighbourhood of w and $gv_i g^{-1}w \to w$ there exists a j such that $gv_j g^{-1}w \in \Omega$. Since $gv_j g^{-1}wy = gv_j y \in X$ the last assertion and the property of Ω imply that $gv_j g^{-1}w \in HV_2$. Also similarly, since $wy = gy \in X, w \in HV_2$. Let $h \in H$ and $v \in V_2$ be such that $w = hv$. Then $gv_j g^{-1}hv \in HV_2$. Since $g \in DV_1 \subset H$ and $v \in V_2$, this implies that $v_j g^{-1}h \in HV_2$. Since $v_j \in V_2 - \{I\}$ and $g^{-1}h \in H,$ by Proposition 4, this implies that $g^{-1}h$ is an upper triangular matrix. Since $g \in DV_1$ this yields that h is an upper triangular matrix. By the restriction on Ω the diagonal entries of $w = hv$ are positive and hence the same holds for h. It is easy to see that an upper triangular matrix with positive entries on the diagonal belongs to H only if it belongs to DV_1. Thus $h \in DV_1$. Therefore $w = hv \in DV$ and hence $gy = wy \in (\Omega \cap DV)y$, as claimed.

Now suppose that there exists an open neighbourhood Ω_1 of I such that $\Omega_1 \subset \Omega, \overline{\Omega}_1$ is compact and $\Omega_1 y \cap DV_1 y \subset (\Omega_1 \cap DV_1)y$. Since X_1 is a minimal nonempty closed DV_1-invariant subset, $\overline{DV_1 y} = X_1$ and in view of this, the last condition readily implies that $\Omega_1 y \cap X_1 \subset (\overline{\Omega}_1 \cap DV_1)y$. But then $(X_1 - DV_1 y)$ is a closed DV_1-invariant subset disjoint from $\Omega_1 y$ and hence by minimality of X_1 as a nonempty closed DV_1-invariant set, we get that $X_1 - DV_1 y$ is empty. As X_1 is DV_1-invariant, this implies that it is a DV_1-orbit. But by Proposition 7 b) there are no closed DV_1-orbits and hence this is a contradiction. Thus there does not exist any neighbourhood Ω_1 as above.

Next let Ω_1 be any open neighbourhood of I such that $\Omega_1 \subset \Omega$ and $\overline{\Omega}_1$ is compact. Then by the above observation there exists $g \in DV_1$ such that $gy \in \Omega_1 y - (\Omega_1 \cap DV_1)y$. Since $\Omega y \cap DV_1 y \subset (\Omega \cap DV)y$ and since any element of Ωy can be expressed uniquely as wy where $w \in \Omega$, the preceding conclusion implies that there exists $w \in (\Omega_1 \cap DV) - DV_1$ such that $gy = wy$. Since this holds for every Ω_1 as above we get that there exist sequences $\{w_i\}$ in $DV - DV_1$ and $\{g_i\}$ in DV_1 such that $w_i \to I$ and $w_i y = g_i y$ for all i. For each i, w_i can be expressed uniquely as $p_i v_2(t_i)$ where $p_i \in DV_1$ and $t_i \in \mathbf{R}$; we see that $t_i \neq 0$ for every i and $t_i \to 0$.

Let $\Delta = \{g \in DV | gy = y\}$. Then Δ is a discrete subgroup of DV and therefore by Lemma 5 it is either contained in V or it is a cyclic subgroup generated by an element of the form vdv^{-1} where $v \in V$ and $d \in D$. It is easy to see that for $v \in V$ and $d \in D$, 0 is an isolated point in the subset $\{t \in \mathbf{R} | DV_1 v_2(t) \text{ contains } vd^j v^{-1} \text{ for some } j \in \mathbf{Z}\}$ of $\mathbf{R}$. For all i we have $g_i^{-1} p_i v_2(t_i) = g_i^{-1} w_i \in \Delta$ with $g_i^{-1} p_i \in DV_1$, $t_i \neq 0$ and $t_i \to 0$ and hence the preceding assertion implies that Δ is not generated by an element of the form vdv^{-1} with $v \in V$ and $d \in D$. Hence Δ is contained in V. Thus $g_i^{-1} p_i v_2(t_i) \in V$ and therefore $g_i^{-1} p_i \in V \cap (DV_1) = V_1$ for all i. Since $g_i^{-1} p_i v_2(t_i) y = y$, this yields that $v_2(t_i) y \in Y$ for all i. Hence $v_2(t_i) Y = v_2(t_i) \overline{V_1 y} = \overline{v_2(t_i) V_1 y} = \overline{V_1 v_2(t_i) y} = \overline{V_1 Y}, = Y$ namely Y is $v_2(t_i)$ invariant, for all i. Since $\{t_i\}$ is contained in $\mathbf{R} - \{0\}$ and $t_i \to 0$ the subgroup generated by $\{t_i | i = 1, 2, \ldots\}$ is dense in $\mathbf{R}$. Hence the preceding assertion implies that $v_2(t) Y = Y$ for all $t \in \mathbf{R}$, namely $V_2 Y = Y$. In particular $V_2 Y$ is contained in X. This completes the proof of the Proposition.

Like Proposition 7, our next proposition also uses, in the general case, Theorem A.8 on the asymptotic behaviour of trajectories of unipotent one-parameter subgroups of G on G/Γ. Also as in the case of that Proposition the proof here goes through without the need for Theorem A.8 if a certain set, namely $\overline{Hg\Gamma}/\Gamma$ as in the statement, is assumed to be *compact* rather than only closed. This observation has some relevance to what one can prove about values of Q, without involving Theorem A.8; we shall amplify this later (see Remark 1).

9. PROPOSITION. *Let $g \in G$ and Q be the quadratic form on $\mathbf{R}^3$ defined by $Q(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$. Suppose that $Hg\Gamma/\Gamma$ is closed. Then there exists $c \neq 0$ such that cQ is a rational quadratic form.*

Proof. As before let S be the space of 3×3 symmetric matrices with real entries. Let $\Delta = (g^{-1}Hg) \cap \Gamma$ and let F be the subspace defined by

$$F = \{\sigma \in S \mid {}^t\delta\sigma\delta = \sigma \quad \text{for all} \quad \delta \in \Delta\}\,.$$

We see that σ_0 is the matrix of the quadratic form Q_0 and hence ${}^th\sigma_0 h = \sigma_0$ for all $h \in H$. Since $g\Delta g^{-1} \subset H$, this implies ${}^tg\sigma_0 g \in F$. In particular F is of positive dimension. Let $\{\xi_1, \cdots, \xi_k\}$ be a basis of F, where $k \geqslant 1$ is the dimension of F. Let $W = S^k = S \oplus S \oplus \cdots \oplus S$ (k copies) and let $\xi \in W$ be the element $(\xi_1, \cdots, \xi_k)$. We define a map $f: G/\Delta \to W$ by $f(x\Delta) = ({}^tx^{-1}\xi_1 x^{-1}, \cdots, {}^tx^{-1}\xi_k x^{-1})$; it is easy to see that the map is well-defined and continuous.

Let $\{u(t)\}$ be a unipotent one-parameter group of matrices contained in H. We assert that there exists a compact subset K of $Hg\Gamma/\Gamma$ such that $\{t \in \mathbf{R} \mid u(t)g\Gamma \in K\}$ is an unbounded subset of $\mathbf{R}$. The assertion is obvious if $Hg\Gamma/\Gamma$ is compact. In the general case it follows from Theorem A.13 in the Appendix (known as Margulis' lemma) and the assumption that $Hg\Gamma/\Gamma$ is a closed subset of G/Γ. Since there is a canonical homeomorphism of $Hg\Gamma/\Gamma$ onto $(g^{-1}Hg)/\Delta$ given by $hg\Gamma \mapsto (g^{-1}hg)\Delta$ for all $h \in H$, the preceding assertion implies that there exists a compact subset K_1 of $(g^{-1}Hg)/\Delta$ such that $\{t \in \mathbf{R} \mid g^{-1}u(t)g\Delta \in K_1\}$ is unbounded; hence the set $R := \{t \in \mathbf{R} \mid f(g^{-1}u(t)g\Delta) \in f(K_1)\}$ is also unbounded. Since f is continuous and K_1 is compact $f(K_1)$ is compact. On the other hand

$$\begin{aligned} &t \mapsto f\big(g^{-1}u(t)g\Delta\big) \\ &= \big({}^tg\,{}^tu(-t)\,{}^tg^{-1}\xi_1 g^{-1}u(-t)g,\ \cdots,\ {}^tg\,{}^tu(-t)\,{}^tg^{-1}\xi_k g^{-1}u(-t)g\big) \end{aligned}$$

is a polynomial map of $\mathbf{R}$ into W. Since R is unbounded and $f(K_1)$ is compact, the map must be a constant map. Thus $f\big(g^{-1}u(t)g\Delta\big) = f(\Delta)$ for all $t \in \mathbf{R}$. Comparing the components we get that ${}^tg\,{}^tu(-t)\,{}^tg^{-1}\xi_j g^{-1}u(-t)g = \xi_j$ for all $j = 1, \cdots, k$ and $t \in \mathbf{R}$.

For each $j = 1, \cdots, k$ put $\eta_j = {}^tg^{-1}\xi_j g^{-1}$. Then by the above observation, for any unipotent one-parameter group of matrices $\{u(t)\}$ contained in H, we have ${}^tu(t)\eta_j u(t) = \eta_j$ for all $j = 1, \cdots, k$ and $t \in \mathbf{R}$. In particular this holds for $\{v_1(t)\}$ and $\{{}^tv_1(t)\}$ in the place of $\{u(t)\}$. But it is easy to see that for $\sigma \in S$ the conditions ${}^tv_1(t)\sigma v_1(t) = \sigma$ and $v_1(t)\sigma\,{}^tv_1(t) = \sigma$ imply that σ is a scalar multiple of σ_0. Since $\eta_1, \cdots, \eta_k$ satisfy these conditions and are linearly independent, this implies that $k = 1$ and $\eta_1 = c\sigma_0$ for some $c \neq 0$. Hence F is the one-dimensional subspace spanned by $\xi_1 = {}^tg\sigma_0 g$.

Since $\Delta \subset \Gamma = SL(3, \mathbf{Z})$ we see that F is the subspace defined by a system of linear equations with integer coefficients; the entries of σ are the variables. As $F \neq \{0\}$, the system of equations has a nonzero solution and hence, the coefficients being integers implies that there exists a nonzero solution in integers. Thus F contains a nonzero integral matrix. Since ${}^t g\sigma_0 g$ spans F we get that there exists $c \neq 0$ such that $c\,{}^t g\sigma_0 g$ is an integral matrix. Since $Q(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$, ${}^t g\sigma_0 g$ is the matrix of the quadratic form Q and therefore the preceding assertion implies that cQ is a rational quadratic form. This proves the proposition.

The above argument to deduce from the one dimensionality of F the rationality of a multiple of ${}^t g\sigma_0 g$, was pointed out by A. Borel. Our earlier argument involved Galois automorphisms. While the two arguments are essentially equivalent, the present form is evidently more suitable.

Before embarking on the proof of the theorem we also note the following simple observation; it would be appropriate to formulate it for all $\mathbf{R}^n$, $n \geqslant 2$.

10. LEMMA. *Let* Ω *be a nonempty open subset of* $\mathbf{R}^n$, *where* $n \geqslant 2$, *such that for all* $w \in \Omega$ *and* $t \geqslant 1$, $tw \in \Omega$. *Then* Ω *contains a primitive integral element.*

Proof. Since Ω is a nonempty open subset of $\mathbf{R}^n$ and $n \geqslant 2$ there exist $p, q \in \Omega$ such that p and q are rational (that is, all their entries are rational), linearly independent and $tp + (1-t)q \in \Omega$ for all $t \in [0, 1]$. By replacing them by the multiples kp and kq where k is a suitable positive integer we may assume that $p, q \in \mathbf{Z}^n$. The condition on Ω as in the hypothesis then implies further that $sp + tq \in \Omega$ for all $s, t \geqslant 0$ such that $s + t \geqslant 1$.

There exists $\gamma \in SL(n, \mathbf{Z})$, namely a $n \times n$ integral matrix with determinant 1, and a natural number m (namely the g.c.d. of the coordinates of p) such that $\gamma p = me_1, e_1, \cdots, e_n$ being the standard basis of $\mathbf{R}^n$. (This follows from [13] Ch. I, §3, Theorem 5, for instance). Let γq be expressed as $m_1e_1 + m_2e_2 + \cdots + m_ne_n$, where $m_1, \cdots, m_n \in \mathbf{Z}$. Since p and q are linearly independent so are γp and γq and hence there exists $i \geqslant 2$ such that $m_i \neq 0$. Let p_0 be a positive prime number such that $p_0 \geqslant m_1$ and p_0 does not divide m_i. Let $r = p_0e_1 + m_2e_2 + \cdots + m_ne_n = (p_0 - m_1)e_1 + \gamma q$. Then as $p_0 \nmid m_i$, r is a primitive integral element and hence so is $\gamma^{-1}r$. But $\gamma^{-1}r = \gamma^{-1}(p_0 - m_1)e_1 + q = m^{-1}(p_0 - m_1)p + q$. Since $p_0 - m_1 \geqslant 0$, by our remark above this shows that $\gamma^{-1}r \in \Omega$. This proves the Lemma.

Proof of the Main Theorem. We begin by noting that it is enough to prove the theorem for $n = 3$; an elementary argument for this simple observation, well known to experts, may be found in the beginning of the proof of Theorem 1 in [7] – we shall not repeat it here. We now consider the case of $n = 3$. Let Q be a nondegenerate indefinite quadratic form on $\mathbf{R}^3$. The matrices of both Q and Q_0 have to be equivalent (cogradient) to one of $d(-1)$ or $-d(-1)$ (cf. [11], Ch. V, Theorem 6 or [21], § 12.5). Hence there exists a nonsingular matrix ρ, say $\rho = \lambda g$ where $\lambda \in \mathbf{R}^*$ and $g \in G$, such that either $Q(p) = Q_0(\rho p) = \lambda^2 Q_0(gp)$ for all $p \in \mathbf{R}^3$ or $Q(p) = -Q_0(\rho p) = -\lambda^2 Q_0(gp)$ for all $p \in \mathbf{R}^3$. In view of this, in proving the theorem, without loss of generality we may assume Q to be the quadratic form defined by $Q(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$, where $g \in G$.

Now let $\mathfrak{p}$ be the set of primitive integral vectors; viz. primitive elements in $\mathbf{Z}^3$. We see that $\Gamma\mathfrak{p} = \mathfrak{p}$, Γ being the subgroup $SL(3, \mathbf{Z})$ as before. Now

$$Q(\mathfrak{p}) = Q_0(g\mathfrak{p}) = Q_0(g\Gamma\mathfrak{p}) = Q_0(Hg\Gamma\mathfrak{p}) .$$

Hence by continuity $Q_0(\overline{Hg\Gamma\mathfrak{p}})$ is contained in $\overline{Q(\mathfrak{p})}$. Since Q is not a multiple of a rational quadratic form, by Proposition 9, $Hg\Gamma/\Gamma$ is not closed. Hence by Proposition 8 there exists $y \in \overline{Hg\Gamma/\Gamma}$ such that either $V_2^+ y$ or $V_2^- y$ is contained in $\overline{Hg\Gamma/\Gamma}$. Suppose that $V_2^+ y$ is contained in $\overline{Hg\Gamma/\Gamma}$. Let $g_0 \in G$ be such that $y = g_0\Gamma$. Then $V_2^+ g_0$ is contained in $\overline{Hg\Gamma}$. Hence $Q_0(V_2^+ g_0\mathfrak{p}) \subset Q_0(\overline{Hg\Gamma\mathfrak{p}}) \subset \overline{Q(\mathfrak{p})}$. We shall show that $Q_0(V_2^+ g_0\mathfrak{p}) = \mathbf{R}$. Let $s \in \mathbf{R}$ be given and let $s_0 = \min\{s, 0\}$. Consider the set $\Omega = \{p \in \mathbf{R}^3 \,|\, Q_0(g_0 p) < s_0 \text{ and } Q_1(g_0 p) > 0\}$. Then Ω satisfies the condition of Lemma 10 and therefore contains a primitive integral element. Thus there exists $p \in \mathfrak{p}$ such that $Q_0(g_0 p) < s_0 \leqslant s$ and $Q_1(g_0 p) > 0$. Let $t = (s - Q_0(g_0 p))/2Q_1(g_0 p)$. Then $t > 0$ and, by (iii), $Q_0(v_2(t) g_0 p) = Q_0(g_0 p) + 2tQ_1(g_0 p) = s$. Thus $s \in Q_0(V_2^+ g_0\mathfrak{p})$. This shows that $Q_0(V_2^+ g_0\mathfrak{p}) = \mathbf{R}$. Hence $\overline{Q(\mathfrak{p})} = \mathbf{R}$ or equivalently $Q(\mathfrak{p})$ is dense in $\mathbf{R}$, as desired. A similar argument works if $V_2^- y$ is contained in $\overline{Hg\Gamma/\Gamma}$. This proves the theorem.

Remark 1. It was noted earlier that while in the general case the proofs of Propositions 7 and 9 (and hence also Proposition 8 as it depends on Proposition 7) involve Theorem A.8, one can do without the latter under certain compactness conditions in each case. Specifically, Propositions 7 and 9 were proved without recourse to Theorem A.8 when X and $Hg\Gamma/\Gamma$ as in their respective statements are compact. Also proving Proposition 8 when the set X as in its statement is compact involves Proposition 7 only for compact subsets. We shall refer to the particular cases of Propositions 7, 8 and 9 with the appropriate set as above assumed to be compact, as the restricted versions of the respective propositions. It may be of some interest to note that one can indeed deduce the following result on values of quadratic forms just from these restricted versions. "Let Q be a nondegenerate indefinite quadratic form on $\mathbf{R}^3$, $n \geqslant 3$, which is not a multiple of a rational quadratic form (just as in the Main Theorem). Then for any $\varepsilon > 0$ there exists a $p \in \mathbf{Z}^n - \{0\}$ such that $|Q(p)| < \varepsilon$". We see this as follows. Firstly, as in the case of the main theorem, this needs to be proved only for $n = 3$ and Q defined by $Q(p) = Q_0(gp)$ for all $p \in \mathbf{R}^3$, where $g \in G$ is fixed. Since Q is not a multiple of a rational form, the restricted version of Proposition 9 implies that $Hg\Gamma/\Gamma$ is not compact. Therefore either $\overline{Hg\Gamma/\Gamma}$ is compact and $Hg\Gamma/\Gamma$ is not closed or $\overline{Hg\Gamma/\Gamma}$ is noncompact. If the former condition holds then the restricted version of Proposition 8 implies that there exists $y \in \overline{Hg\Gamma/\Gamma}$ such that either $V_2^+ y$ or $V_2^- y$ is contained $\overline{Hg\Gamma/\Gamma}$ and then the proof can be completed just like that of the Main theorem. Now suppose that $\overline{Hg\Gamma/\Gamma}$ is noncompact. Then by the Mahler criterion (cf. [13] Ch. 3, Theorem 2 or [2] Ch. V – see also the following Appendix for some details) there exist sequences $\{h_i\}$ in H and $\{p_i\}$ in $\mathbf{Z}^3 - \{0\}$ such that $h_i g p_i \to 0$. Then $Q(p_i) = Q_0(gp_i) = Q_0(h_i g p_i) \to 0$ and hence, given $\varepsilon > 0$ there exists $p = p_i$ for some i such that $|Q(p)| < \varepsilon$; this proves the claim.

The above assertion which is the same as Theorem 1 of [20] proves the Oppenheim conjecture for the quadratic forms for which there does not exist any $p \in \mathbf{Z}^n - \{0\}$ such that $Q(p) = 0$. For the general case some more work is needed (cf. Theorem 1′ in [20]). Using Theorem A.8 not only takes care of this difficulty but enables one to get a primitive integral solution.

Remark 2. The study of orbits of unipotent one-parameter subgroups in [7] and [8] also leads to some more results on values of quadratic forms, than the Main theorem here. One of these, involving the quadratic form and also the corresponding bilinear form has already been mentioned in the introduction (see (ii)). In [8] we also prove the following. Let Q and Q' be two quadratic forms on $\mathbf{R}^3$ such that no nonzero linear combination of Q and Q' is a rational quadratic form. Suppose that there exists a basis f_1, f_2, f_3 of $\mathbf{R}^3$ such that

$$\begin{aligned} Q(p_1 f_1 + p_2 f_2 + p_3 f_3) &= 2p_1 p_3 - p_2^2 \\ \text{and} \quad Q'(p_1 f_1 + p_2 f_2 + p_3 f_3) &= p_3^2 \end{aligned}$$

for all $p_1, p_2, p_3 \in \mathbf{R}$. Then for any $a, b \in \mathbf{R}, b > 0$, and $\varepsilon > 0$ there exists a primitive integral point p such that

$$|Q(p) - a| < \varepsilon \quad \text{and} \quad |Q'(p) - b| < \varepsilon .$$

As yet it does not seem that these results would be accessible by elementary arguments.

The study of flows on homogeneous spaces leads also to various other number theoretic results, which we shall not go into here. We refer the reader to the survey articles [4] and [19] for some of the ideas involved.

APPENDIX

TRAJECTORIES OF UNIPOTENT FLOWS AND MINIMAL SETS

We prove here a 'qualitative version' of Theorem 1.1 of [7] and use it to deduce the general case of Proposition 7. We also deduce a result used in the proof of Proposition 9. The proof of the 'qualitative version', namely Theorem A.1 below is in the same spirit at that of Theorem 2.1 of [7] and the earlier related results in [16], [3] and [5]. But the exposition here is simpler, especially on account of the weaker formulation.

We begin by setting up some notation. As before we denote by $\mathbf{R}^n$, $n \geqslant 2$, the n-dimensional vector space of n-rowed column vectors with entries in $\mathbf{R}$, by $e_1, \cdots, e_n$ the standard basis of $\mathbf{R}^n$ and by $\mathbf{Z}^n$ the subgroup generated by $\{e_1, \cdots, e_n\}$. By a lattice in $\mathbf{R}^n$ we mean a subgroup generated by n linearly independent elements in $\mathbf{R}^n$; a discrete subgroup Δ of $\mathbf{R}^n$ is a lattice if and only if $\mathbf{R}^n/\Delta$ is compact. (Cf. [13], Ch. I, §3, Theorem 2.)

We equip $\mathbf{R}^n$ with the usual inner product $< \, , \, >$ with $e_1, \cdots, e_n$ as an orthonormal basis, and the corresponding norm $\| \cdot \|$. This induces an inner product on each (vector) subspace of $\mathbf{R}^n$. For any subgroup Δ of $\mathbf{R}^n$ we denote by $\Delta_{\mathbf{R}}$ the subspace of $\mathbf{R}$ spanned by Δ. Let Δ be a discrete subgroup of $\mathbf{R}^n$. Then there exists a basis $x_1, \cdots, x_r$, where $r =$ dimension of $\Delta_{\mathbf{R}}$, such that Δ is generated by $\{x_1, \cdots, x_r\}$ (cf. [13], Ch. I, §3, Theorem 2). Let τ be a linear transformation of $\Delta_{\mathbf{R}}$ such that $\tau^{-1}x_1, \cdots, \tau^{-1}x_r$ is an orthonormal basis of $\Delta_{\mathbf{R}}$, with respect to the induced inner product. The number $|\det \tau|$ is independent of the choice of the basis $x_1, \cdots, x_r$ and the linear transformation τ, so long as the above conditions are satisfied; the number is called the determinant of Δ and is denoted by $d(\Delta)$.

As usual let $SL(n, \mathbf{R})$ be the group of $n \times n$ matrices with entries in $\mathbf{R}$ and determinant 1. By a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$ we mean a unipotent one-parameter group of $n \times n$ matrices (-they are clearly contained in $SL(n, \mathbf{R})$.) We now state the theorem on orbits of lattices under unipotent one-parameter subgroups, needed in the proofs of Propositions 7 and 9.

A.1. THEOREM. *Let $n \geqslant 2$ be fixed. Then for $\sigma > 0$ there exists a $\delta > 0$ such that for any lattice Λ in $\mathbf{R}^n$, any unipotent one-parameter subgroup $\{u_t\}_{t \in \mathbf{R}}$ of $SL(n, \mathbf{R})$ and any $T \geqslant 0$ either there exists $s \geqslant T$ such that $\| u_s x \| \geqslant \delta$ for all $x \in \Lambda - \{0\}$ or there exists a nonzero (discrete) subgroup Δ of Λ such that $d(u_t \Delta) < \sigma$ for all $t \in [0, T]$.*

We introduce some more notation and prove some preliminary results before going to the proof of the theorem. For any lattice Λ in $\mathbf{R}^n$ we denote by $\mathscr{S}(\Lambda)$ the set of all nonzero subgroups of the form $\Lambda \cap W$, where W is a (vector) subspace of $\mathbf{R}^n$; such a subgroup is called a complete subgroup of Λ. For each lattice Λ we equip $\mathscr{S}(\Lambda)$ with the partial order given by the inclusion relation on subgroups and for any totally ordered subset S of $\mathscr{S}(\Lambda)$ define

$$\mathscr{C}(S, \Lambda) = \{\Delta \in \mathscr{S}(\Lambda) - S \mid S \cup \{\Delta\} \text{ is a totally ordered subset}\} ;$$

the subgroups belonging to $\mathscr{C}(S, \Lambda)$ are said to be compatible with S.

We next observe some properties of the function d on class of discrete subgroups of $\mathbf{R}^n$. It is easy to see that if Δ is a discrete subgroup generated by r linearly independent elements $x_1, \cdots, x_r$ then the determinant of the $r \times r$ matrix $(< x_i, x_j >)$ (with $< x_i, x_j >$ in the i th row and j th column) is $d^2(\Delta)$. Under the same conditions, $d^2(\Delta)$ also coincides with the sum of squares of the determinants of all $r \times r$ minors of the $n \times r$ matrix with $x_1, \cdots, x_r$ as its columns. This may be verified either directly or using exterior products (if the reader would wish to save trouble, it may be mentioned here that Propositions 7 and 9 involve the contents of the Appendix and in particular these observations only for $n = 3$). These characterisations enable us to deduce various properties of d needed in the sequel.

A.2. LEMMA. a) *For any lattice Λ in $\mathbf{R}^n$ and any $\rho > 0$ the set $\{\Delta \in \mathscr{S}(\Lambda) \mid d(\Lambda) < \rho\}$ is finite.*

b) *Let Δ be a discrete subgroup of $\mathbf{R}^n$. Let $x \in \mathbf{R}^n - \Delta_{\mathbf{R}}$ and let Δ' be the (discrete) subgroup generated by Δ and x. Then $d(\Delta') \leqslant \| x \| \, d(\Delta)$.*

Proof. a) Clearly, for any nonsingular matrix g there exist constants a and b such that for any discrete subgroup Δ, $ad(\Delta) \leqslant d(g\Delta) \leqslant bd(\Delta)$. Since any lattice is of the form $g\mathbf{Z}^n$ for some nonsingular matrix g, this shows that it is enough to prove a) for $\Lambda = \mathbf{Z}^n$. If Δ is a subgroup of $\mathbf{Z}^n$ generated by r linearly independent elements $x_1, \cdots, x_r$, then the determinants of all $r \times r$ minors of the $n \times r$ matrix with columns $x_1, \cdots, x_r$ are integers. The condition $d(\Delta) < \rho$ then implies, by one of the characterisations of d, that there are only finitely many possibilities for the values of the determinants of the minors. The finiteness assertion in the Lemma therefore follows from the fact that if the corresponding $r \times r$ minors of two $n \times r$ matrices ξ and η have same determinants then the columns of ξ and η span the same subspace of $\mathbf{R}^n$.

ii) This is obvious, for instance, from the characterisation of $d(\Delta)$ in terms of the determinants of $r \times r$ minors of the $n \times r$ matrix whose columns are linearly independent and generate Δ.

A.3. LEMMA. *Let Δ be a nonzero discrete subgroup of $\mathbf{R}^n$ and let $\{u_t\}$ be a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$. Then $d^2(u_t\Delta)$ is a polynomial in t of degree at most $2n(n-1)$. Further, $d(u_t\Delta)$ is constant (that is, $d(u_t\Delta) = d(\Delta)$ for all $t \in \mathbf{R}$) if and only if $\Delta_{\mathbf{R}}$ is $\{u_t\}$-invariant (that is $u_t\Delta_{\mathbf{R}} = \Delta_{\mathbf{R}}$ for all $t \in \mathbf{R}$).*

Proof. If ν is a $n \times n$ nilpotent matrix then by the Jordan canonical form $\nu^n = 0$. This implies that for any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n, \mathbf{R})$ and any $x \in \mathbf{R}^n$, the coordinates (entries) of $u_t x$ are polynomials in t of degree at most $n - 1$. Now let Δ be a discrete subgroup generated by r linearly independent elements $x_1, \cdots, x_r$. Then $d^2(u_t\Delta)$ is the determinant of the $r \times r$ matrix $(< u_t x_i, u_t x_j >)$. By the preceding remark each entry $< u_t x_i, u_t x_j >$ is a polynomial in t of degree at most $2(n-1)$. Hence the determinant is a polynomial of degree at most $2n(n-1)$.

Next let Δ be a discrete subgroup such that $d(u_t\Delta) = d(\Delta)$ for all $t \in \mathbf{R}$. Let $x_1, \cdots, x_r$ be linearly independent elements generating Δ. The determinant of each $r \times r$ minor of the $n \times r$ matrix with columns $u_t x_1, \cdots, u_t x_r$ is a polynomial in t. Since sum of squares of these is $d^2(u_t\Delta) = d^2(\Delta)$ for all $t \in \mathbf{R}$, it follows that each of them is constant. Thus for each $t \in \mathbf{R}$ any $r \times r$ minor of the $n \times r$ matrix with columns $u_t x_1, \cdots, u_t x_r$ has the same determinant as the corresponding minor in the $n \times r$ matrix with columns $x_1, \cdots, x_r$. This implies that for any t, $u_t x_1, \cdots, u_t x_r$ span the same subspace as $x_1, \cdots, x_r$, or equivalently $u_t\Delta_{\mathbf{R}} = \Delta_{\mathbf{R}}$. This proves the Lemma.

For any $m \in \mathbf{N}$ we denote by $\mathscr{P}_m$ the set of all nonnegative polynomials of degree at most m; 'nonnegative' refers to the values being nonnegative — some of the coefficients could be negative. For the proof of Theorem 8 we need the following simple properties of nonnegative polynomials.

A.4. LEMMA. a) *For any $m \in \mathbf{N}$ and $\lambda > 1$ there exists $\varepsilon > 0$ such that the following holds: if $P \in \mathscr{P}_m$ and there exists $s \in [0, 1]$ such that $P(s) \geqslant 1$ and $P(1) < \varepsilon$ then there exists $t \in [1, \lambda]$ such that $P(t) = \varepsilon$.*

b) *For any* $m \in \mathbf{N}$ *and* $\mu > 1$ *there exist constants* $\varepsilon_1, \varepsilon_2 > 0$ *such that the following holds: if* $P \in \mathscr{P}_m, P(s) \leqslant 1$ *for all* $s \in [0,1]$ *and* $P(1) = 1$ *then there exists* $i, 0 \leqslant i \leqslant m,$ *such that* $\varepsilon_1 \leqslant P(t) \leqslant \varepsilon_2$ *for all* $t \in [\mu^{2i+1}, \mu^{2i+2}]$.

Proof. It can be seen that given an interval I of positive length and a $c > 0$ there exists a constant M such that any $P \in \mathscr{P}_m$ such that $P(t) \leqslant c$ for all $t \in I$, has all the coefficients of absolute value at most M; in particular, any sequence of polynomials bounded by c on I has a subsequence converging to a polynomial in $\mathscr{P}_m$. Now if a) does not hold there must exist a sequence $\{P_k\}$ in $\mathscr{P}_m$ such that $P_k(t) \to 0$ uniformly on $[1, \lambda]$ but the supremum of each P_k on $[0, 1]$ is at least 1; this is impossible by the above observation. To prove b) we first observe that existence of the upper bound ε_2 follows from the bound on the coefficients as above, when we take $I = [0, 1]$ and $c = 1$. Thus if b) does not hold there exists a sequence $\{P_k\}$ in $\mathscr{P}_m$ such that for each $k, P_k(s) \leqslant 1$ for all $s \in [0,1], P_k(1) = 1$ and $\inf\{P_k(t) \mid t \in [\mu^{2i+1}, \mu^{2i+2}]\} \to 0$ as $k \to \infty$, for each $i = 0, \cdots, m$; this is impossible since the limit of any subsequence would be a nontrivial polynomial in $\mathscr{P}_m$ with at least $m + 1$ zeros.

For the rest of the argument we fix some constants as follows: Let $n \in \mathbf{N}$ and $\mu > 1$ be arbitrary. Let $m = 2n^2$ and $\lambda > 1$ be such that $(\lambda - 1) \leqslant (\mu - 1)/\mu^{2m+2}$. Let $0 < \alpha < 1$ be such that condition a) as in Lemma A.4 holds for $\varepsilon = \alpha^2$ with m and λ as above and let $0 < \beta_1 < 1 < \beta_2$ be such that condition b) of Lemma A.4 holds for $\varepsilon_1 = \beta_1^2$ and $\varepsilon_2 = \beta_2^2$ with m and μ as above.

A.5. PROPOSITION. *Let* $\{u_t\}$ *be a unipotent one-parameter subgroup of* $SL(n, \mathbf{R}), \Lambda$ *be a lattice in* $\mathbf{R}^n$ *and* S *be a totally ordered subset of* $\mathscr{S}(\Lambda)$. *Let* $\tau > 0$ *and* $T \geqslant 0$ *be such that for each* $\Phi \in \mathscr{L}(S, \Lambda)$ *there exists a* $t \in [0, T]$ *such that* $d(u_t\Phi) \geqslant \tau$. *Then either* $d(u_T\Phi) \geqslant \alpha\tau$ *for all* $\Phi \in \mathscr{L}(S, \Lambda)$ *or there exist a* $\Delta \in \mathscr{L}(S, \Lambda)$ *and a* $T_1 \in [T, (2 - \mu^{-1})T]$ *such that the following conditions are satisfied:*

i) $\tau\alpha\beta_1 \leqslant d(u_t\Delta) \leqslant \tau\alpha\beta_2$ *for all* $t \in [T_1, T + \mu(T_1 - T)]$

ii) *for each* $\Phi \in \mathscr{L}(S, \Lambda)$ *there exists* $t \in [T, T_1]$ *such that* $d(u_t\Phi) \geqslant \alpha\tau$.

Proof. Let $\mathscr{F} = \{\Phi \in \mathscr{L}(S, \Lambda) \mid d(u_T\Phi) < \alpha\tau\}$. If $\mathscr{F}$ is empty then we are through. Now suppose that $\mathscr{F}$ is nonempty. By Lemma A.2 a) $\mathscr{F}$ is finite; say $\mathscr{F} = \{\Phi_1, \cdots, \Phi_q\}$, where $q \geqslant 1$. For each $j, 1 \leqslant j \leqslant q$, we choose

$t_j \in [T, \lambda T]$ as follows: Observe that $d(u_T \Phi_j) < \alpha\tau$ and that there exists, by hypothesis, a $t \in [0, T]$ such that $d(u_t \Phi_j) \geqslant \tau$. Hence applying Lemma A.4, a) to the polynomial $t \mapsto d^2(u_{tT}\Phi_j)/\tau^2$ we conclude that there exists a $t_j \in [T, \lambda T]$ such that $d(u_{t_j}\Phi_j) = \alpha\tau$; taking the smallest such number we may also assume t_j to have the further property that $d(u_t \Phi_j) \leqslant \alpha\tau$ for all $t \in [T, t_j]$.

Next let $1 \leqslant k \leqslant q$ be such that $t_j \leqslant t_k$ for all $1 \leqslant j \leqslant q$. We choose $\Delta = \Phi_k$. Then we have $d(u_t \Delta) \leqslant \alpha\tau$ for all $t \in [T, t_k]$ and $d(u_{t_k}\Delta) = \alpha\tau$. Hence by Lemma A.4 b), applied to the polynomial $t \mapsto d^2(u_{(t_k - T)t + T}\,\Delta)/\alpha^2\tau^2$, it follows that there exists an i such that $0 \leqslant i \leqslant m$ and

$$(*) \qquad \tau\alpha\beta_1 \leqslant d(u_t \Delta) \leqslant \tau\alpha\beta_2 \quad \text{for all} \quad t \in [T_1, T_2]\ ,$$

where $T_1 = T + \mu^{2i+1}(t_k - T)$ and $T_2 = T + \mu^{2i+2}(t_k - T)$. Then

$$\begin{aligned} T + \mu(T_1 - T) \ = \ T + \mu^{2i+2}(t_k - T) \ &\leqslant \ T + \mu^{2m+2}(t_k - T) \\ &\leqslant \ T + \mu^{2m+2}(\lambda - 1)T \ \leqslant \ \mu T\,, \end{aligned}$$

since $i \leqslant m$, $t_k \in [T, \lambda T]$ and $(\lambda - 1) \leqslant (\mu - 1)/\mu^{2m+2}$. This shows that $T_1 \in [T, (2 - \mu^{-1})T]$. Also $(*)$ shows that condition i) as in the Proposition is satisfied for Δ. Condition ii) is obvious from the construction; if $\Phi \notin \mathscr{F}$ then $d(u_T \Phi) \geqslant \alpha\tau$ and if $\Phi \in \mathscr{F}$, say $\Phi = \Phi_j$ where $1 \leqslant j \leqslant q$, then we have $T \leqslant t_j \leqslant t_k \leqslant T_1$ and $d(u_{t_j}\Phi_j) = \alpha\tau$, which verifies the condition for all $\Phi \in \mathscr{L}(S, \Lambda)$. Hence the Proposition.

A.6. COROLLARY. *Let* $\{u_t\}, \Lambda, S, \tau > 0$ *and* $T \geqslant 0$ *be as in Proposition A.5. Let* p *be the cardinality of* S. *Then there exist a totally ordered subset* M *of* $\mathscr{S}(\Lambda)$ *containing* S *and a* $R \in [T, \mu T]$ *such that the following conditions are satisfied:*

1) $\alpha^{(n-p)}\beta_1\tau \leqslant d(u_R \Phi) \leqslant \alpha\beta_2\tau$ *for all* $\Phi \in M - S$

2) $d(u_R \Phi) \geqslant \alpha^{(n-p)}\tau$ *for all* $\Phi \in \mathscr{L}(M, \Lambda)$.

Proof. We proceed by induction on $(n-p)$. If $p = n$ then S is a maximal totally ordered subset (so $\mathscr{L}(S, \Lambda)$ is empty) and the desired assertion holds for $M = S$. We now assume the result for $p + 1$ in the place of p and consider Λ, S, τ and T as in the hypothesis. If $d(u_T \Phi) \geqslant \alpha\tau$ for $\Phi \in \mathscr{L}(S, \Lambda)$ then we can choose $M = S$ and $R = T$. If not, then by Proposition A.5 there exist $\Delta \in \mathscr{L}(S, \Lambda)$ and $T_1 \in [T, (2 - \mu^{-1})T]$ such that $\tau\alpha\beta_1 \leqslant d(u_t \Delta) \leqslant \tau\alpha\beta_2$ for all $t \in [T_1, T + \mu(T_1 - T)]$ and for each $\Phi \in \mathscr{L}(S, \Lambda)$ there exists a $t \in [T, T_1]$

such that $d(u_t\Phi) \geqslant \alpha\tau$. Put $\Lambda_1 = u_T\Lambda$, $S_1 = \{u_T\Phi \mid \Phi = \Delta$ or $\Phi \in S\}$ and $\tau_1 = \alpha\tau$. Then Λ_1 is a lattice in $\mathbf{R}^n$, S_1 is a totally ordered subset of $\mathscr{S}(\Lambda_1)$ and the second part of the preceding conclusion implies that the hypothesis of the corollary applies to Λ_1, S_1, τ_1 and $T_1 - T$ in the place of Λ, S, τ and T respectively; we note that any $\Psi \in \mathscr{L}(S_1, \Lambda_1)$ is of the form $u_T\Phi$, $\Phi \in \mathscr{L}(S, \Lambda)$. Hence by the induction hypothesis there exist a subset M_1 of $\mathscr{S}(\Lambda_1)$ containing S_1 and a $R_1 \in [T_1 - T, \mu(T_1 - T)]$ such that $\alpha^{(n-p-1)}\beta_1\tau_1 \leqslant d(u_{R_1}\Delta_1) \leqslant \alpha\beta_2\tau_1$ for all $\Delta_1 \in M_1 - S_1$ and $d(u_{R_1}\Phi) \geqslant \alpha^{(n-p-1)}\tau_1$ for all $\Phi \in \mathscr{L}(M_1, \Lambda_1)$. Put $M = \{u_{-T}\Delta_1 \mid \Delta_1 \in M_1\}$ and $R = T + R_1$. Then $T \leqslant R \leqslant T + \mu(T_1 - T) \leqslant \mu T$, since $T_1 \in [T, (2 - \mu^{-1})T]$. Observe that $M - S = \{\Phi \mid \Phi = \Delta$ or $u_T\Phi \in M_1 - S_1\}$. The choice of Δ, using Proposition A.5 shows that Condition 1) in the conclusion of the Corollary holds for $\Phi = \Delta$. If $u_T\Phi \in M_1 - S_1$ then we have $d(u_R\Phi) = d(u_{R_1}u_T\Phi) \in [\alpha^{(n-p-1)}\beta_1\tau_1, \alpha\beta_2\tau_1] \subset [\alpha^{(n-p)}\beta_1\tau, \alpha\beta_2\tau]$, since $\tau_1 = \alpha\tau$ and $\alpha < 1$. Thus Condition 1) holds for all $\Phi \in M - S$. For $\Phi \in \mathscr{L}(M, \Lambda)$ we have $d(u_R\Phi) = d(u_{R_1}u_T\Phi) \geqslant \alpha^{(n-p-1)}\tau_1 = \alpha^{(n-p)}\tau$, since $u_T\Phi \in \mathscr{L}(M, \Lambda_1)$ and $\tau_1 = \alpha\tau$; this shows that Condition 2) is also satisfied. This proves the Corollary.

Proof of Theorem A.1. Let n and σ be as in the hypothesis of the theorem. Let $\mu > 1$ be chosen arbitrarily and let α, β_1 and β_2 be the constants chosen ahead of Proposition A.5, depending on n and μ; recall that $0 < \alpha < 1$ and $0 < \beta_1 < 1 < \beta_2$. Let $t = \min\{\sigma, \sigma^{-1}\}$ and let $\delta = \alpha^n\beta_1\beta_2^{-1}\tau$.

Now let $\{u_t\}$ be any unipotent one-parameter subgroup of $SL(n, \mathbf{R})$, Λ be any lattice in $\mathbf{R}^n$ and let $T \geqslant 0$ be such that there does not exist any nonzero subgroup Δ of Λ such that $d(u_t\Delta) < \sigma$ for all $t \in [0, T]$. This implies that for all $\Phi \in \mathscr{S}(\Lambda)$ there exists $\tau \in [0, T]$ such that $d(u_t\Phi) \geqslant \sigma \geqslant \tau$. In other words, the condition in the Corollary holds if we choose S to be the empty subset. Hence by the Corollay there exists a totally ordered subset M of $\mathscr{S}(\Lambda)$ and a $R \in [T, \mu T]$ such that $\alpha^n\beta_1\tau \leqslant d(u_R\Phi) \leqslant \alpha\beta_2\tau \leqslant \beta_2$ for all $\Phi \in M$ and $d(u_R\Phi) \geqslant \alpha^n\tau$ for all $\Phi \in \mathscr{L}(M, \Lambda)$. Now let x be any primitive element in Λ and let Δ be the subgroup generated by x. Then $\Delta \in \mathscr{S}(\Lambda)$. If x is contained in every element of M then we see that $\Delta \in M \cup \mathscr{L}(M, \Lambda)$ and hence $\| u_R x \| = d(u_R\Delta) \geqslant \alpha^n\beta_1\tau \geqslant \delta$. Now suppose that x is not contained in some elements of M and let Φ be the largest element of M not containing x. Let Ψ be the smallest complete subgroup of Λ (element of $\mathscr{S}(\Lambda)$) containing Φ and x. Then we see that $\Psi \in M \cup \mathscr{L}(M, \Lambda)$, as every element of M containing Φ as a proper subgroup also contains x. Now, by Lemma A.2 b) $d(u_R\Psi) \leqslant \| u_R x \| \, d(u_R\Phi)$. But since $\Phi \in M$ and $\Psi \in M \cup \mathscr{L}(M, \Lambda)$ we have

$d(u_R\Phi) \leqslant \beta_2$ and $d(u_R\Psi) \geqslant \alpha^n\beta_1\tau$. Thus we get that $\| u_R x \| \geqslant \alpha^n\beta_1\beta_2^{-1}\tau = \delta$. Hence $\| u_R x \| \geqslant \delta$ for all primitive x in Λ and hence the same holds for all $x \in \Lambda - \{0\}$, thus proving the Theorem.

A.7. COROLLARY. *Given* $\sigma > 0$ *there exists a neighbourhood* Ω *of* 0 *in* $\mathbf{R}^n$ *such that for any unipotent one-parameter subgroup* $\{u_t\}$ *in* $SL(n, \mathbf{R})$ *and any lattice* Λ *in* $\mathbf{R}^n$ *one of the following holds:*

1) $\{t \geqslant 0 \mid u_t\Lambda \cap \Omega = (0)\}$ *is an unbounded subset of* $\mathbf{R}$.

2) *there exists a nonzero subgroup* Δ *of* Λ *such that the subspace spanned by* Δ *is* $\{u_t\}$*-invariant and* $d(u_t\Delta) = d(\Delta) < \sigma$ *for all* $t \in \mathbf{R}$.

Proof. Let $\delta > 0$ be such that Theorem A.1 holds for the given σ and let $\Omega = \{x \in \mathbf{R}^n \mid \| x \| < \delta\}$. Let $\{u_t\}$ and Λ be as in the hypothesis and suppose that Condition 1) does not hold. Then by Theorem A.1 there exists a nonzero subgroup Δ of Λ such that $d(u_t\Delta) < \sigma$ for all $t \geqslant 0$. Since $d^2(u_t\Delta)$ is a polynomial in t, this implies that $d(u_t\Delta)$ is constant; i.e. $d(u_t\Delta) = d(\Delta) < \sigma$ for all $t \in \mathbf{R}$. By Lemma A.3, this implies that the subspace $\Delta_{\mathbf{R}}$ spanned by Δ is $\{u_t\}$-invariant. This proves the corollary.

We next relate Theorem A.1 and Corollary A.7 to behaviour of orbits of unipotent one-parameter groups of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, where $SL(n, \mathbf{Z})$ is the subgroup consisting of integral matrices. This involves the Mahler criterion (sometimes also called Mahler's selection theorem) recalled below. The reader may refer [2], [13] or [24] depending on the background; one could also consult Mahlers original paper [15].

Let $\mathscr{L}_n$ be the set of all lattices in $\mathbf{R}^n$. On $\mathscr{L}_n$ one defines a topology by prescribing that for each basis $x_1, \cdots, x_n$ of $\mathbf{R}^n$ and $\varepsilon > 0$ the set $\Omega(x_1, \cdots, x_n, \varepsilon)$, of all lattices Λ such that Λ is generated by a basis $y_1, \cdots, y_n$ of $\mathbf{R}^n$ satisfying $\| x_i - y_i \| < \varepsilon$ for all i, be open. This indeed defines a first countable Hausdorff topology on $\mathscr{L}_n$. The Mahler criterion asserts that if $\{\Lambda_i\}$ is a sequence in $\mathscr{L}_n$ and there exist c and δ such that for all i, $d(\Lambda_i) \leqslant c$ and $\| x \| \geqslant \delta$ for all $x \in \Lambda_i - \{0\}$ then $\{\Lambda_i\}$ has a convergent subsequence. The criterion implies in particular that $\mathscr{L}_n$ is locally compact.

Now let $\mathscr{U}_n$ be the subset of $\mathscr{L}_n$ consisting of all lattices of determinant 1. Then $\mathscr{U}_n$ is a closed subset, as d is continuous, and in particular it is locally compact. For each $g \in SL(n, \mathbf{R})$ and $\Lambda \in \mathscr{U}_n$, $g\Lambda \in \mathscr{U}_n$ and the map $(g, \Lambda) \mapsto g\Lambda$ defines a continuous action of $SL(n, \mathbf{R})$ on $\mathscr{U}_n$. It is easy to see that the action is transitive and that $SL(n, \mathbf{Z})$ is the isotropy subgroup of the lattice $\mathbf{Z}^n$, under the action. Hence $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, equipped with the quotient topology, is homeomorphic to $\mathscr{U}_n$ via the correspondence

$gSL(n, \mathbf{Z}) \mapsto g\mathbf{Z}^n$ for $g \in SL(n, \mathbf{R})$ (cf. [9], Ch. V, §1, Theorem 8 or [10], (1.6.1)). The Mahler criterion therefore implies that for any $\delta > 0$ the set

$$\{gSL(n, \mathbf{Z}) \mid \| gp \| \geqslant \delta \quad \text{for all} \quad p \in \mathbf{Z}^n - \{0\}\}$$

is a compact subset of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. Theorem A.1 and Corollary A.7 therefore imply the following

A.8. THEOREM. *Let $n \geqslant 2$ be fixed. Then for any $\sigma > 0$ there exists a compact subset K of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ such that for any $x = gSL(n, \mathbf{Z}) \in SL(n, \mathbf{R})/SL(n, \mathbf{Z})$, where $g \in G$, and any unipotent one-parameter subgroup $\{u_t\}$ of $SL(n, \mathbf{R})$ the following conditions are satisfied:*

a) *for any $T \geqslant 0$ either there exists a $t \geqslant T$ such that $u_t x \in K$ or there exists a nonzero discrete subgroup Δ of $\mathbf{Z}^n$ such that $d(u_t g\Delta) < \sigma$ for all $t \in [0, T]$,*

b) *if $\{t \geqslant 0 \mid u_t x \in K\}$ is bounded then there exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that the subspace spanned by Δ is $\{g^{-1}u_t g\}$-invariant and $d(u_t g\Delta) = d(g\Delta) < \sigma$ for all $t \in \mathbf{R}$.*

We next deduce the general case of Proposition 7, which we had deferred until proving the above theorem. We follow the notation G, Γ, V_1, DV_1 etc., as in the main part. The diagonal matrix diag $(\lambda, 1, \lambda^{-1})$ where $\lambda \in \mathbf{R}^*$ will be denoted by $a(\lambda)$, rather than $d(\lambda)$, to avoid confusion with $d(\Delta)$ for discrete subgroups Δ. Also as before we denote by e_1, e_2, e_3 the standard basis of $\mathbf{R}^3$. The subspaces spanned by $\{e_1\}$ and $\{e_1, e_2\}$ are denoted by W_1 and W_2 respectively.

We first prove part b) of Proposition 7, namely the following:

A.9. PROPOSITION. *There are no closed DV_1-orbits. Any nonempty closed DV_1-invariant subset contains a minimal nonempty closed DV_1-invariant subset.*

Proof. Let K be a compact subset of G/Γ such that the contention of Theorem A.8 holds for ($n = 3$ and) $\sigma = 1$. We first show that for any $x = g\Gamma \in G/\Gamma$, where $g \in G$, there exists $\lambda_0 > 0$ such that for all $\lambda \geqslant \lambda_0$, $\{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded. Let $g \in G$ be given and let $x = g\Gamma$. Define

$$\lambda_0 = \max\{1, 1/d(g\mathbf{Z}^3 \cap W_1), 1/d(g\mathbf{Z}^3 \cap W_2)\} .$$

Let $\lambda \geqslant \lambda_0$ be arbitrary. Let Δ be a nonzero discrete subgroup $\mathbf{Z}^3$ such that $\Delta_{\mathbf{R}}$ is a proper subspace invariant under the action of $g^{-1}a(\lambda)^{-1}V_1 a(\lambda)g = g^{-1}V_1 g$. Then $g\Delta_{\mathbf{R}}$ is a nonzero proper V_1-invariant subspace. A simple computation shows that W_1 and W_2 are the only such subspaces. Hence $g\Delta_{\mathbf{R}} = W_1$ or W_2. Both W_1 and W_2 are $a(\lambda)$-invariant and the determinant of the restriction of $a(\lambda)$ to either subspace is λ. Hence the preceding observation implies that $d\big(a(\lambda)g\Delta\big) = \lambda d(g\Delta)$. Since $g\Delta$ is contained in either $g\mathbf{Z}^3 \cap W_1$ or $g\mathbf{Z}^3 \cap W_2$, by the choice of λ_0 we get that $d(g\Delta) \geqslant \lambda_0^{-1}$. Hence $d\big(a(\lambda)g\Delta\big) \geqslant \lambda/\lambda_0 \geqslant 1 = \sigma$. In view of this verification for all Δ as above, Theorem A.8 b) implies that $\{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded as claimed; note that as $\sigma = 1$, the subgroup Δ in Theorem A.8 b) spans a proper subspace.

We now deduce the assertions as in the proposition. If possible let $x \in G/\Gamma$ be such that DV_1x is a closed orbit in G/Γ. Let $\Phi = \{g \in G \mid gx = x\}$. Then Φ is a discrete subgroup of DV_1 and the map $\theta\colon DV_1/\Phi \to DV_1x$ defined by $\theta(g\Phi) = gx$ for all $g \in DV_1$ is a homeomorphism (cf. [9], Ch. V, §1, Theorem 8 or [10], (1.6.1)). By Lemma 6 Φ is either contained in V_1 or it is a cyclic subgroup generated by an element of the form vdv^{-1} where $d \in D$ and $v \in V_1$. Suppose the latter possibility holds. Then we see that for each $\lambda > 0$, $V_1 a(\lambda)\Phi$ is closed and $t \mapsto v_1(t)a(\lambda)\Phi$ defines a homeomorphism of $\mathbf{R}$ onto $V_1 a(\lambda)\Phi/\Phi$. Since θ is a homeomorphism, this implies that for each $\lambda > 0$, $V_1 a(\lambda)x$ is closed and $t \mapsto v_1(t)a(\lambda)x$ is a homeomorphism of $\mathbf{R}$ onto $V_1 a(\lambda)x$. But, by our observation above, there exists λ_0 such that for $\lambda \geqslant \lambda_0$, $\{t \geqslant 0 \mid v_1(t)a(\lambda)x \in K\}$ is unbounded. This is a contradiction since by the preceding observation it implies that $\{v_1(t)a(\lambda)x \mid t \geqslant 0\} \cap K$ is a closed noncompact subset of K. Now suppose Φ is contained in V_1. Let $\{\lambda_i\}$ be a sequence of positive numbers such that $\lambda_i \to \infty$. Then we see that as $\Phi \subset V_1$, for any sequence $\{t_i\}$ in $\mathbf{R}$, $\{a(\lambda_i)v_1(t_i)\Phi\}$ has no convergent subsequence in DV_1/Φ. Since θ is a homeomorphism this implies that for any sequence $\{t_i\}$ in $\mathbf{R}$, $\{a(\lambda_i)v_1(t_i)x\}$ has no convergent subsequence. But this is a contradiction since K is compact and for all large λ there exists $t \geqslant 0$ such that $v_1(t)a(\lambda)x = a(\lambda)\big(v_1(\lambda^{-1}t)\big)x \in K$. Hence there are no closed DV_1-orbits.

Now let X be any nonempty closed DV_1-invariant subset of G/Γ. We see that if $\{X_i\}_{i\in I}$ is a totally ordered family (with respect to inclusion) of nonempty closed DV_1-invariant subsets of X (indexed by a set I), then $\cap_{i\in I} X_i$ is nonempty as it contains $\cap_{i\in I}(X_i \cap K)$ and by the above observation each $X_i \cap K$ is a nonempty compact subset. Hence by Zorn's lemma the class of all nonempty closed DV_1-invariant subsets of X has a minimal element. This proves the Proposition.

To prove the other part of Proposition 7 we need the following Lemmas.

A.10. LEMMA. *Let* $q = 1$ *or* 2 *and for any* $\rho > 0$ *let*

$$A(q, \rho) = \{g\Gamma \mid g \in G, g\mathbf{Z}^3 \cap W_q \quad \textit{spans} \quad W_q \quad \textit{and} \quad d(g\mathbf{Z}^3 \cap W_q) = \rho\} .$$

Then $A(q, \rho)$ *is a closed subset of* G/Γ.

Proof. It is straightforward to verify that any subset as in the statement can be expressed as $Q_q a\Gamma/\Gamma$ for some diagonal matrix a, Q_1 and Q_2 being the subgroups defined by

$$Q_1 = \{g \in G \mid ge_1 = e_1\} \quad \text{and} \quad Q_2 = \{g \in G \mid {}^t ge_3 = e_3\} .$$

Now consider the natural action of G on $\mathbf{R}^3$. We see that Γe_1 is a discrete subset of $\mathbf{R}^3$. Hence so is $\Gamma s e_1$ for any $s \in \mathbf{R}$. Let b be a diagonal matrix. Then $be_1 = se_1$ for some $s \in \mathbf{R}$ and hence Γbe_1 is a closed subset of $\mathbf{R}^3$. The continuity of the action and the fact that Q_1 is the subgroup consisting of all elements fixing e_1 now implies that $\Gamma b Q_1$ is a closed subset of G, for any diagonal matrix b. Hence so is $Q_1 a\Gamma = (\Gamma a^{-1} Q_1)^{-1}$, for any diagonal matrix a. This proves the case of the Lemma with $q = 1$. The case of $q = 2$ follows from a similar argument with the contragradient action, defined by $(g, p) \mapsto {}^t g^{-1} p$ for all $p \in \mathbf{R}^3$, in the place of the natural action, and e_3 in the place of e_1.

A.11. LEMMA. *Let* Z *be a locally compact space and let* $\{\varphi_t\}_{t \in R}$ *be a one-parameter group of homeomorphisms of* Z *acting continuously on* Z. *Suppose that there exists a compact subset* K *of* Z *such that for each* $z \in Z$, *the sets* $\{t \geqslant 0 \mid \varphi_t z \in K\}$ *and* $\{t \leqslant 0 \mid \varphi_t z \in K\}$ *are unbounded. Then* Z *is compact.*

Proof. Let $\varphi = \varphi_1$. Replacing K by the larger compact set $\{\varphi_s z \mid -1 \leqslant s \leqslant 1, z \in K\}$ if necessary, we may assume that for each $z \in Z$, $\{k \in \mathbf{N} \mid \varphi^k z \in K\}$ and $\{k \in \mathbf{N} \mid \varphi^{-k} z \in K\}$ are unbounded subsets of $\mathbf{N}$. Let K_1 be a compact neighbourhood of K and let $\Omega = Z - K_1$. Let $B = \cap_{j=0}^{\infty} \varphi^j \overline{\Omega}$. Then $\varphi^{-j} B \subset B \subset \overline{\Omega} \subset Z - K$ for all $j \in \mathbf{N}$ and hence the condition on K implies that B is empty. Hence φB is empty. Since K_1 is compact this implies that there exists $m \in \mathbf{N}$ such that $\cap_{j=1}^{m} \varphi^i \overline{\Omega}$ is contained in Ω. Then $\cap_{j=0}^{m} \varphi^j \Omega = \cap_{j=1}^{m} \varphi^j \Omega = E$ say. Then we see that $\varphi E \subset E$ and hence

$\varphi^j E \subset E$ for all $j \in \mathbf{N}$. Since $E \subset \Omega \subset Z - K$, the condition on K implies that E is empty. Hence $Z = \cup_{j=1}^{m} \varphi^j(Z-\Omega)$, which is compact.

Part a) of Proposition 7 now follows from the following Proposition and the earlier observation for compact invariant sets.

A.12. PROPOSITION. *Any nonempty closed V_1-invariant subset of G/Γ contains a compact nonempty V_1-invariant subset.*

Proof. Let X be a nonempty closed V_1-invariant subset of G/Γ. For $q = 1, 2$ and any $\rho > 0$ let $A(q, \rho)$ denote the closed subset of G/Γ as in Lemma A.10. In proving the Proposition, by replacing X by a smaller (nonempty) subset if necessary, we may assume that for each $q = 1, 2$ and $\rho > 0$, either $X \cap A(q, \rho) = \emptyset$ or $X \subset A(q, \rho)$; note that the sets $A(q, \rho)$ are V_1-invariant and that for each q the sets $\{A(q, \rho)\}_{\rho > 0}$ are mutually disjoint. Now let $\sigma \leqslant 1$ be such that if X is contained in $A(q, \rho)$ for some $q = 1$ or 2 and $\rho > 0$ then $\sigma \leqslant \rho$. Let K be a compact subset of G/Γ such that the contention of Theorem A.8 holds for this σ. We shall show that for each $x \in X$ the sets $\{t \geqslant 0 | v_1(t)x \in K\}$ and $\{t \leqslant 0 | v_1(t)x \in K\}$ are unbounded; by Lemma A.11 this implies that X (rather the replaced set) is compact, thus proving the proposition. Suppose for some $x \in X$, say $x = g\Gamma$ where $g \in G$, one of the sets as above is bounded. Then by Theorem A.8, applied to either $\{v_1(t)\}$ or $\{v_1(-t)\}$ in the place of $\{u_t\}$ and x as above, it follows that there exists a nonzero subgroup Δ of $\mathbf{Z}^n$ such that $\Delta_{\mathbf{R}}$ is $g^{-1}V_1 g$-invariant and $d(v_1(t)g\Delta) = d(g\Delta) < \sigma$ for all $t \in \mathbf{R}$. Since $\sigma \leqslant 1$ (as in the proof of Proposition A.7) we see that $g\Delta_{\mathbf{R}} = W_1$ or W_2. This implies that $x = g\Gamma \in X \cap A(q, \rho)$, where $q = 1$ or 2 and ρ is the determinant of the complete subgroup of Λ containing $g\Delta$ and spanning the same subspace. By the assumption on X we now get that $X \subset A(q, \rho)$. By our choice of σ we then have $\sigma \leqslant \rho$. But this is a contradiction since $\rho \leqslant d(g\Delta) < \sigma$. Hence the sets as above are unbounded and thus the proof is complete.

As noted earlier Propositions A.12 and A.9 yield parts a) and b) of Proposition 7, which thus stands proved. We next note the following variation of Theorem A.8, first proved by Margulis [16], which was used in the proof of Proposition 9.

A.13. THEOREM. *Let $n \geqslant 2$ be fixed. Let $\{u_t\}$ be a unipotent one-parameter subgroup of $SL(n, \mathbf{R})$ and let $x \in SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. Then there exists a compact subset K of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$ such that $\{t \geqslant 0 | u_t x \in K\}$ is an unbounded subset of $\mathbf{R}$.*

Proof. Let $g \in G$ such that $x = gSL(n, \mathbf{Z})$ and let $\Lambda = g\mathbf{Z}^n$. In view of Lemma A.2 a) there exists $\sigma > 0$ such that $d(\Delta) > \sigma$ for all subgroups Δ of Λ. Hence by Theorem A.1 there exists $\delta > 0$ such that for any $T \geqslant 0$ there exists a $s \geqslant T$ for which $\| u_s \xi \| \geqslant \delta$ for all $\xi \in \Lambda - \{0\}$. Let $K = \{hSL(n, \mathbf{Z}) | \| hp \| \geqslant \delta$ for all $p \in \mathbf{Z}^n - \{0\}\}$. Then by the Mahler criterion, recalled earlier, K is a compact subset of $SL(n, \mathbf{R})/SL(n, \mathbf{Z})$. From the choices it is clear that $\{s \geqslant 0 | u_s x \in K\}$ is an unbounded subset. This proves the theorem.

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(Reçu le 16 octobre 1989)

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