§1. Introduction

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ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES

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§ 1. INTRODUCTION

In this article we study the extrinsic inversive geometry of curves in the Euclidean plane $\mathbf{R}^2$ under the inversive group $G = PSL_2(\mathbf{C})^{\sim}$ of *general* Möbius transformations. This is $PSL_2(\mathbf{C})$ extended by complex conjugation. $PSL_2(\mathbf{C})$ itself is the *special,* or orientation preserving Möbius transformations. An introduction to classical inversive geometry may be found in [18].

As our model for this geometry we take the complex plane $\mathbf{C}$ (with coordinate $z = x + iy$) together with the point at infinity, ∞. The underlying topological space is of course S^2 and G is the group of conformal and anticonformal transformations of S^2, but we use the standard Euclidean metric on $\mathbf{C}$. We shall assume that all our curves are oriented and smooth.

In § 2 we recall Coxeter's invariant (cf. [5]), the "inversive distance", between two non-intersecting circles. This is the imaginary part of their imaginary angle of intersection. Based on this idea we obtain a proof of a result of Kneser (cf. [9], p. 48) which says that on a vertex-free part of a curve γ the osculating circles never intersect. Using the square root of the inversive distance between neighbouring osculating circles on γ we obtain an invariant 1-form ω (the infinitesimal inversive arc-length). This 1-form was apparently first discovered by H. Liebmann in 1923 [12], although the name of G. Pick is also mentioned by Blaschke in [2]. If γ is parametrized by the arc-length s and if $\kappa(s)$ denotes the curvature at the point $\gamma(s)$, then the 1-form ω can be identified as the 1-form $\sqrt{|\kappa'(s)|}\, ds$ (cf. our § 2, or [3], p. 92), and can be extended continuously over the vertices. It follows that the set of vertices (points where $\kappa'(s) = 0$) of a curve is invariant under the inversive group. The integral of this invariant 1-form gives the inversive arc-length, $\nu = \int \omega$, a

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natural invariant parameter for curves in inversive geometry. We end the section with a table for the inversive arc-length for various conics.

The classical four vertex theorem, due to Mukhopadhaya in 1909, states that every simple closed curve in $\mathbf{R}^2$ has at least four vertices. Though the standard proof is easy in the case of convex curves, Kneser's 1911 [11] generalization to the non-convex case is strangely more complicated, and the result is usually stated without proof in introductory texts. Simple and elegant proofs have been given by Valette in 1957 [17] (cf. also Pinkall 1987 [15]) and Osserman in 1985 [14]. The theorem is also known to be true for S^2 but the usual proof is again quite complicated. Furthermore it is easy to construct simple closed curves on the torus with only two vertices. In § 3 we present a simple new proof of the four vertex theorem for (not necessarily convex) simple closed curves on $\mathbf{R}^2$ based on the conformal invariance of the vertices. The moral is that the four vertex theorem is really a theorem in inversive differential geometry, where the larger symmetry group is a powerful aid. In §4 we consider a generalization of the form ω to curves γ on an arbitrary Riemannian surface given by the formula:

$$\omega_\gamma = \sqrt{|\kappa_g'|}\, ds$$

where κ_g is the geodesic curvature of the curve on the surface. It turns out that this form is invariant under maps between surfaces which preserve the curves of constant geodesic curvature, the so-called "concircular maps". As a consequence of this we show in §5 the following result.

THEOREM. *If* γ *is a smooth, null-homotopic, simple closed curve on a complete Riemannian surface* M *of constant curvature, then the geodesic curvature of* γ *has at least four local extrema.*

The remainder of the paper continues a general study of curves in the inversive plane. The method used throughout is the method of moving frames in one of its simpler incarnations, systematically developed by A. Tresse [16] called "the method of reduced equations". In fact the spirit here is much the same as the first part of É. Cartan's beautiful book [4].

In §6 we show that for each non-vertex point p on a curve γ there is a *unique* orientation preserving Möbius transformation $g \in G$ such that $g^{-1}(p) = 0$ and the Taylor expansion for the curve $g^{-1}(\gamma)$ at the origin has the normal form

$$y = \pm\frac{x^3}{6} + Q\frac{x^5}{60} + O(x^6) \tag{1.1}$$

where $\pm = \operatorname{sgn}(\kappa')$. The denominator 60 (rather than the seemingly more natural $5! = 120$) represents a normalization of Q to simplify formula 1.3 below and the calculations for the loxodrome in §9. It is clear that Q is invariant under (special) Möbius transformations and so we call it *the inversive curvature of γ at p.* It can be calculated in terms of the Euclidean curvature $\kappa(s)$ and its derivatives with respect to Euclidean arc-length by means of the formula

$$Q = \frac{4(\kappa''' - \kappa^2\kappa')\kappa' - 5\kappa''^2}{8\kappa'^3} \tag{1.2}$$

We note that although the sign of Q depends on the orientation of the plane, it is nevertheless independent of the orientation of the curve. The curvature Q corresponds to the invariant $b/2$ which Blaschke ([2], end of §21) obtains by a completely different (and roundabout) method).

The procedure described above gives rise to a *Frenet lift* $g:\gamma - \{\text{vertices}\} \to G$, which is a curve on the Lie group G parametrized by inversive arc-length. In §7 we show that parallel translation of the tangent vector $dg/dv \in T_g(G)$ back to the identity by g^{-1} yields the formula

$$g^{-1}\frac{dg}{dv} = \begin{pmatrix} 0 & 1 \\ \frac{1}{2}\operatorname{sgn}(\kappa')(Q-i) & 0 \end{pmatrix} \tag{1.3}$$

It follows that the curvature Q *determines* the vertex-free curve up to a Möbius transformation.

The curves with Q constant are especially interesting as they constitute the "lines and circles" of inversive geometry. These are studied in §9 and turn out to be what Blaschke [2] calls "loxodromes"; that is, they are the equiangular spirals (Bernouli's *spira mirabilis*) and their inversive images. Loxodromes are the orbits of 1-parameter subgroups of loxodromic transformations.

In §10 we use a simple notion of *contact* to define and determine the complex of smooth, local "geometric" differential forms Λ^*_{geo} on a vertex free curve in $\mathbf{R}^1$. This is a universal complex equipped with a homomorphism $\Psi_\gamma : \Lambda^*_{geo} \to \Lambda^*(\gamma)$ to the de Rham complex of γ for every vertex free curve γ, and satisfying the invariance property that $\Psi_\gamma = g^*\Psi_{g(\gamma)}$ for every $g \in G$. It turns out that Λ^*_{geo} is generated by the function Q and the form ω so that these are essentially the *only* interesting smooth local invariants of curves in $\mathbf{R}^2$.