

**Zeitschrift:** L'Enseignement Mathématique  
**Herausgeber:** Commission Internationale de l'Enseignement Mathématique  
**Band:** 36 (1990)  
**Heft:** 1-2: L'ENSEIGNEMENT MATHÉMATIQUE

**Artikel:** ON THE INVERSIVE DIFFERENTIAL GEOMETRY OF PLANE CURVES  
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**Kapitel:** §7. The canonical map  $g:G$   
**DOI:** <https://doi.org/10.5169/seals-57907>

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The stabilizer of the expression  $y = x^4/24 + O(x^6)$  has order 2 and is generated by

$$z \mapsto -\bar{z}.$$

It follows that the non-degeneracy of a vertex is an invariant of inversive geometry.

§7. THE CANONICAL MAP  $g: \gamma \rightarrow G$

The considerations of the last section allow us to define a canonical map  $g_\gamma: \gamma \rightarrow G$  for vertex free curves  $\gamma$  by mapping a point  $p \in \gamma$  to  $g_\gamma(p) \in G$ , which is the unique group element such that  $g_\gamma(p)^{-1}$  sends  $p$  to the origin and  $g_\gamma(p)^{-1}(\gamma)$  has oriented contact of order 4 with the standard curve  $y = x^3/6$  at the origin. We note that if  $\gamma' = h(\gamma)$  for some  $h \in G$ , then obviously  $g_{\gamma'}(h(p)) = h(g_\gamma(p))$ . Of course altering the initial choice of the origin and the axes used there to describe the model will alter  $g_\gamma$ , but only by right multiplication by some fixed element of  $G$ . If  $\sigma: (\alpha, \beta) \rightarrow \mathbf{C}$  is a parametrization of the curve by Euclidean arc-length  $s$ , and  $\sigma'(s) = e^{i\theta(s)}$ , then the curvature of the curve at  $\sigma(s)$  is  $\theta'(s) = \kappa(s)$ , and we have the following explicit formula for  $g$ .

$$g(s) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\kappa'' - 2i\kappa\kappa')/4\kappa' & 1 \end{pmatrix} \begin{pmatrix} |\kappa'|^{-1/4} & 0 \\ 0 & |\kappa'|^{1/4} \end{pmatrix}$$

The first two factors are Euclidean motions whose inverse puts  $\gamma$  into oriented first order contact with the oriented  $x$ -axis. The rest improve the order of contact to 4 as in §6. It is convenient to regard  $g$  as a function of the inverse arc-length  $v$ . Now  $g(v)$  is a curve on the Lie group  $G$ , with tangent vector  $dg/dv$  at  $g(v)$ . Left translation by  $g(v)^{-1}$  moves this tangent vector to the origin to yield

$$(7.1) \quad c(v) = g(v)^{-1} \frac{dg}{dv}$$

which is a vector in the Lie algebra  $sl_2(\mathbf{C})$  of 2 by 2 complex matrices of trace zero. As  $v$  varies  $c(v)$  inscribes a curve on this Lie algebra. Indeed it is well known (e.g. [13], p. 71) that this curve determines the original curve  $g(v)$  up to left translation by an arbitrary constant element of  $G$ . Here is an explicit formula for the curve  $c(v)$ . It is easy but rather tedious to verify it.

$$c(v) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where } T = \frac{1}{2} \operatorname{sgn}(\kappa') (Q - i)$$

and  $Q$  is as in §6. It follows that the inversive curvature  $Q$  determines the curve up to an orientation preserving inversive automorphism.

### §8. RELATION WITH CARTAN'S MOVING FRAMES

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succinctly in [7]. The canonical line bundle

$$p: \xi \rightarrow \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} & & \\ \downarrow & p \downarrow & \\ z = \frac{z_1}{z_2} \leftrightarrow [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) & & \end{array}$$

Let  $\sigma: (\alpha, \beta) \rightarrow \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$  be a curve; we choose an arbitrary lift  $\hat{\sigma} = (z_1(t), z_2(t))$  and set  $f_1 = \lambda \hat{\sigma}$ ,  $f_2 = \dot{f}_1 = \lambda(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$ , where  $\dot{\phantom{x}} = \frac{d}{dt}$ . Thus  $(f_1, f_2)$  is a frame in  $\mathbf{C}^2$ . We try to choose  $\lambda$  so that this frame has area 1. The condition on  $\lambda$  is:

$$\begin{aligned} 1 &= \operatorname{Area}(f_1, f_2) = \operatorname{Area}(\lambda(z_1(t), z_2(t)), \lambda(\dot{z}_1, \dot{z}_2)) \\ &= \lambda^2(z_1 \dot{z}_2 - z_2 \dot{z}_1), \quad \text{or } 1 = -(\lambda z_2)^2 \dot{z} \end{aligned}$$

Thus  $\lambda = \frac{i}{z_2 \sqrt{\dot{z}}}$  will do, and we have

$$\begin{aligned} f_1 &= \frac{i}{\sqrt{\dot{z}}} (z, 1), \\ \text{and } f_2 &= \dot{f}_1 = -\frac{1}{2} i \ddot{z} \dot{z}^{-3/2} (z, 1) + i \dot{z}^{-1/2} (z, 0). \end{aligned}$$

Finally a calculation shows that  $\dot{f}_2 = S f_1$ , where  $S = \frac{3}{4} \ddot{z}^2 \dot{z}^{-2} - \frac{1}{2} \ddot{z} \dot{z}^{-1}$ .

Of course  $S$  is the *Schwartzian derivative* which this calculation interprets as a "curvature" of  $\sigma$ . Now the Schwartzian  $S$  depends on the particular parametrization which is used for the curve. For our purposes we wish to use