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The stabilizer of the expression $y = x^4/24 + O(x^6)$ has order 2 and is generated by

$$z \mapsto -\bar{z}\,.$$

It follows that the non-degeneracy of a vertex is an invariant of inversive geometry.

§7. The canonical map $g:\gamma \to G$

The considerations of the last section allow us to define a canonical map $g_\gamma:\gamma \to G$ for vertex free curves γ by mapping a point $p \in \gamma$ to $g_\gamma(p) \in G$, which is the unique group element such that $g_\gamma(p)^{-1}$ sends p to the origin and $g_\gamma(p)^{-1}(\gamma)$ has oriented contact of order 4 with the standard curve $y = x^3/6$ at the origin. We note that if $\gamma' = h(\gamma)$ for some $h \in G$, then obviously $g_{\gamma'}(h(p)) = h(g_\gamma(p))$. Of course altering the initial choice of the origin and the axes used there to describe the model will alter g_γ, but only by right multiplication by some fixed element of G. If $\sigma:(\alpha, \beta) \to \mathbf{C}$ is a parametrization of the curve by Euclidean arc-length s, and $\sigma'(s) = e^{i\theta(s)}$, then the curvature of the curve at $\sigma(s)$ is $\theta'(s) = \kappa(s)$, and we have the following explicit formula for g.

$$g(s) = \begin{pmatrix} 1 & \sigma \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ (\kappa'' - 2i\kappa\kappa')/4\kappa' & 1 \end{pmatrix} \begin{pmatrix} |\kappa'|^{-1/4} & 0 \\ 0 & |\kappa'|^{1/4} \end{pmatrix}$$

The first two factors are Euclidean motions whose inverse puts γ into oriented first order contact with the oriented x-axis. The rest improve the order of contact to 4 as in §6. It is convenient to regard g as a function of the inverse arc-length ν. Now $g(\nu)$ is a curve on the Lie group G, with tangent vector $dg/d\nu$ at $g(\nu)$. Left translation by $g(\nu)^{-1}$ moves this tangent vector to the origin to yield

$$c(\nu) = g(\nu)^{-1}\,\frac{dg}{d\nu} \tag{7.1}$$

which is a vector in the Lie algebra $sl_2(\mathbf{C})$ of 2 by 2 complex matrices of trace zero. As ν varies $c(\nu)$ inscribes a curve on this Lie algebra. Indeed it is well known (e.g. [13], p. 71) that this curve determines the original curve $g(\nu)$ up to left translation by an arbitrary constant element of G. Here is an explicit formula for the curve $c(\nu)$. It is easy but rather tedious to verify it.

$$c(\nu) = \begin{pmatrix} 0 & 1 \\ T & 0 \end{pmatrix}, \quad \text{where} \quad T = \frac{1}{2}\operatorname{sgn}(\kappa')\,(Q-i)$$

and Q is as in §6. It follows that the inversive curvature Q determines the curve up to an orientation preserving inversive automorphism.

§8. Relation with Cartan's moving frames

Let us sketch a more usual way of obtaining a Frenet lift. The connection with the Schwartzian described here can be found, for example, in Cartan's book [4] and very succintly in [7]. The canonical line bundle

$$p : \xi \to \mathbf{P}^1(\mathbf{C})$$

has a pedestrian description (away from the zero-section) as:

$$\begin{array}{ccc} & (z_1, z_2) \in \xi - \{\text{zero section}\} = \mathbf{C}^2 - \{0\} \\ & \updownarrow \quad\quad {}_p\downarrow \\ z = \dfrac{z_1}{z_2} \leftrightarrow & [z_1, z_2] \in \mathbf{P}^1(\mathbf{C}) \end{array}$$

Let $\sigma : (\alpha, \beta) \to \mathbf{R}^2 \subset \mathbf{P}^1(\mathbf{C})$ be a curve; we choose an arbitrary lift $\hat{\sigma} = (z_1(t), z_2(t))$ and set $f_1 = \lambda\hat{\sigma}, f_2 = \dot{f}_1 = \dot{\lambda}(z_1, z_2) + \lambda(\dot{z}_1, \dot{z}_2)$, where $\cdot = \dfrac{d}{dt}$. Thus (f_1, f_2) is a frame in $\mathbf{C}^2$. We try to choose λ so that this frame has area 1. The condition on λ is:

$$\begin{aligned} 1 = \operatorname{Area}(f_1, f_2) &= \operatorname{Area}\big(\lambda(z_1(t), z_2(t)),\ \lambda(\dot{z}_1, \dot{z}_2)\big) \\ &= \lambda^2(z_1\dot{z}_2 - z_2\dot{z}_1), \text{ or } 1 = -(\lambda z_2)^2\dot{z}\ . \end{aligned}$$

Thus $\lambda = \dfrac{i}{z_2\sqrt{\dot{z}}}$ will do, and we have

$$f_1 = \frac{i}{\sqrt{\dot{z}}}\,(z, 1)\ ,$$

$$\text{and} \quad f_2 = \dot{f}_1 = -\frac{1}{2}\,i\,\ddot{z}\,\dot{z}^{-3/2}(z, 1) + i\,\dot{z}^{-1/2}(\dot{z}, 0)\ .$$

Finally a calculation shows that $\dot{f}_2 = S f_1$, where $S = \dfrac{3}{4}\,\ddot{z}^2\dot{z}^{-2} - \dfrac{1}{2}\,\dddot{z}\,\dot{z}^{-1}$.

Of course S is the *Schwartzian derivative* which this calculation interprets as a "curvature" of σ. Now the Schwartzian S depends on the particular parametrization which is used for the curve. For our purposes we wish to use