CONTACT GEOMETRY AND WAVE PROPAGATION

L'Enseignement Mathématique, t. 36 (1990), p. 215-266

CONTACT GEOMETRY AND WAVE PROPAGATION [1]

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Dedicated to the memory of O. P. Shcherbak

§ 1. BASIC DEFINITIONS

Symplectic geometry is at present generally accepted as the natural basis for mechanics and for the calculus of variations.

Contact geometry, which is the odd-dimensional counterpart of the symplectic one, is not yet so popular, although it is the natural basis for optics and for the theory of wave propagation.

The relations between symplectic and contact geometries are similar to those between linear algebra and projective geometry. First, the two related theories are formally more or less equivalent: every theorem in symplectic geometry may be formulated as a contact geometry theorem, and any assertion in contact geometry may be translated into the language of symplectic geometry. Next, all the calculations look algebraically simpler in the symplectic case, but geometrically things are usually better understood when translated into the language of contact geometry.

Hence one is advised to calculate symplectically but to think rather in contact geometry terms.

Finally, most of the global, topological results are more natural in the contact gometry context, and so we can up-date the well known slogan "projective geometry is all geometry" by saying "contact geometry is all geometry". For instance, most of the facts of the differential geometry of submanifolds of Euclidean or of Riemannian space may be translated into the language of contact geometry and may be proved in this more general setting. Thus we can use the intuition of Euclidean or Riemannian geometry to guess the general results of contact geometry, whose applications to the problem of

[1]) Survey lectures given at the University of Oxford in November and December 1988 under the sponsorship of the International Mathematical Union.

This article has already been published in *Monographie de l'Enseignement Mathématique*, N° 34, Université de Genève, 1989.

ordinary differential geometry provide new information in this classical domain.

Definition. A *contact structure* on an odd-dimensional manifold is a field of tangent hyperplanes, which is generic at every point.

These hyperplanes are called the *contact hyperplanes* (of a given contact structure).

By a theorem of Darboux (that we shall prove later) all the generic fields of tangent hyperplanes are locally diffeomorphic to each other. Hence one may consider one particular example of a contact structure on a manifold of a given dimension, and substitute the genericity condition by the condition of being equivalent to the chosen normal form.

Such a normal form may be described as the field of hyperplanes given by $\alpha = 0$, where

$$\alpha = dy - pdq \tag{1}$$

(here $y, p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ are called the Darboux coordinates).

The nondegeneracy condition for the field $\alpha = 0$ is $\alpha \wedge (d\alpha)^n = 0$. Or, equivalently, $d\alpha$ defines a bilinear symplectic structure on each contact hyperplane $\alpha = 0$. Instead of the normal form (1) one uses also such forms as

$$\alpha = dz + (pdq - qdp)/2 . \tag{2}$$

These coordinates are also called Darboux coordinates (sometimes the 2 is dropped or some of the signs are inversed).

Example 1. Let us consider the set of 1-jets of functions $f\colon V^n \to \mathbf{R}$.

Let $(q_1, \ldots, q_n)$ be local coordinates on $V = V^n$ and y be the coordinate in $\mathbf{R}$. A 1-jet of a function $y = f(q)$ is defined by the value of the function and of its first partial derivatives at a given point. Hence the manifold of all 1-jets $J^1(V, \mathbf{R})$ is of dimension $2n + 1$, and its local coordinates are $(p_1, \ldots, p_n;\ q_1, \ldots, q_n; y)$, where $p_k = \partial f / \partial q_k$ (in PDE the coordinates (p, q, y) are usually denoted by (ξ, x, u)).

The manifold of 1-jets is equipped with a *natural contact structure* defined locally by the equation

$$dy = pdq .$$

This contact structure is independent of the particular choice of the coordinates and hence is defined globally. To see this, let us associate to any function f

its 1-graph, which is the set of its 1-jets at all the points of V. The tangent spaces to the 1-graphs of all the functions at any given point of $J^1(V, \mathbf{R})$ belong to a hyperplane in the tangent space to J^1 at that point. This hyperplane's local equation is $dz = pdq$. Hence this equation defines a hyperplane independent of any coordinate system: it admits an intrinsic definition, given above.

This contact structure of the manifold of 1-jets is called its canonical (or natural) contact structure.

Example 2. A *contact element* on a given manifold is a hyperplane in a tangent space. The set of all contact elements on a given manifold $B = B^m$ is fibered over B, and the fibre over a point of B is the projectivized cotangent space of B at this point (called the point of contact).

This set of all contact elements of B is called the space of the *projectivized cotangent bundle* PT^*B. Its dimension $2m - 1$ is odd and it carries a natural contact structure.

This structure is defined by the following construction. A velocity of motion of a contact element is called admissible if the velocity of the point of contact belongs to the contact element. It is easy to see that the admissible velocities form a hyperplane at any given point of PT^*B, and that these hyperplanes define a contact structure.

The set of all the contact elements, tangent to any particular submanifold of B, is an integral manifold of this contact structure of PT^*B. The dimension of such integral manifolds is independent of the dimensions of the original submanifold: it is always $m - 1$, that is almost one half of the dimension of the whole contact manifold.

The integral submanifolds of this maximal dimension of a contact structure are called *Legendre manifolds.*

Thus to every submanifold of the base manifold B there corresponds a Legendre submanifold of the contact elements' manifold PT^*B. For instance, let us start with a point of B (a 0-dimensional submanifold). The corresponding Legendre manifold is the fibre of the bundle $PT^*B \to B$. Hence these fibres are Legendre submanifolds. A fibration (or a foliation) of a contact manifold whose fibres (leaves) are Legendre submanifolds is called a Legendre fibration (foliation). Thus the projectivized cotangent bundle is a Legendre fibration. Another example is the natural fibration $J^1(V, \mathbf{R}) \to J^0(V, \mathbf{R})$ which is the "forgetting of the derivatives" mapping.

At the other extremity we have the hypersurfaces of B. In this case the Legendre submanifold consists of the tangent spaces of the hypersurface. It is naturally diffeomorphic to the hypersurface.

Example 3. Let us consider the particular case PT^*P^n (the base space is the projective space). In this particular case there exists a natural isomorphism

$$PT^*P^n \approx PT^*(P^{n*})$$

where P^{n*} is the dual projective space of P^n. This isomorphism associates to a contact element of P^n (that is to a projective hyperplane and to one of its points) the dual contact element of P^{n*} (consisting of the projective hyperplane considered as a point of P^{n*}, and of the point of P^n, considered as a hyperplane in P^{n*}).

Thus our manifold of contact elements of the projective space is equipped with two contact structures: the first is that of PT^*P^n, the other comes from PT^*P^{n*}.

These two contact structures coincide. (This is a non-trivial theorem having an easy geometrical proof; I leave to the reader the pleasure of discovering it.)

Now let us consider a smooth hypersurface in P^n. Its tangent contact elements form a Legendre submanifold L in PT^*P^n. The fibration $PT^*P^n \to P^{n*}$ is a Legendre fibration. The image of our Legendre submanifold L in P^{n*} is called the *dual hypersurface* of the initial one.

We see that the dual hypersurface of a smooth hypersurface is the image of the corresponding Legendre submanifold of PT^*P^n under the Legendre projection $PT^*P^n \to P^{n*}$.

The image of a Legendre submanifold under a Legendre projection is called the *front* (of the corresponding Legendre submanifold). Hence the dual hypersurface is the front of the Legendre submanifold of tangent hyperplanes of the initial hypersurface.

The affine version of this projective construction is called the Legendre transformation. More precisely, if the initial hypersurface is given by the equation $z = f(q)$, and the dual one by $w = g(p)$, then the function g is called the Legendre transformation of f.

The triple $L \to E \to B$ where the left arrow is the embedding of a Legendre submanifold and the second is a Legendre fibration is called a Legendre projection. A germ of a Legendre projection at a point is called a Legendre singularity.

An equivalence of two Legendre projections (or of two Legendre singularities) is a commutative diagram

$$\begin{array}{ccccc} L_1 & \to & E_1 & \to & B_1 \\ \downarrow & & \downarrow & & \downarrow \\ L_2 & \to & E_2 & \to & B_2 \end{array}$$

whose vertical maps are diffeomorphisms, such that the middle vertical map preserves the contact structure.

Equivalent Legendre projections (or singularities) have diffeomorphic fronts.

Example 4. Let us consider a hypersurface in a Riemannian manifold. The equidistant hypersurfaces are the fronts of the appropriate Legendre mappings.

To see this let us consider the geodesic flow of the oriented contact elements. At time t the element contact point moves at distance t along the geodesic, orthogonal to the element, in the direction defined by the co-orientation. And the moving contact element is always orthogonal to the geodesic.

The geodesic flow of contact elements of B is a one-parameter family of diffeomorphisms of the manifold ST^*B of co-oriented contact elements.

THEOREM. *The geodesic flow of contact elements preserves the natural contact structure of the manifold of contact elements.*

This non-trivial geometrical theorem is one of the formulations of Huygens' principle, which describes the moving wavefront as the envelope of the spherical fronts issuing from the points of the initial front.

The diffeomorphisms of a contact manifold preserving contact structures are called contactomorphisms. They form the contactomorphism group of the manifold — one of the (pseudo) groups of E. Cartan's list of series of simple infinite dimensional groups (the other series are the diffeomorphism groups, the symplectomorphism groups, the biholomorphism group and so on).

Remark. Locally any contact structure is defined as the field of zeroes of an 1-form; any such form is called a *contact form* (associated to a given contact structure). Sometimes one can choose such a form globally (this is the case, for instance, for the 1-jet space's natural contact structure).

In such cases one is tempted to consider the group of diffeomorphisms, preserving the contact form (which is only a subgroup of the true group of contactomorphisms). Some bad people require contact transformations to be elements of this subgroup. This is a mistake to avoid: the subgroup is not intrinsically related to the contact structure, it depends on the particular choice of the contact form.

Example 5. The *pedal hypersurface* of a given hypersurface in the Euclidean space is formed by the points where the tangent hyperplanes meet the perpendiculars issuing from the origin.

The pedal hypersurfaces of smooth hypersurfaces have singularities. Some experimentation shows that they are (generically) the same as singularities of (generic) equidistants or that of hypersurfaces dual to the (generic) smooth ones of the fronts of the generic Legendre submanifolds.

For instance, in the case of plane curves the generic front singularities are just cusp points of order 3/2 (the front being locally diffeomorphic to the semicubical parabola $x^2 = y^3$). The generic singularities of the front surfaces in the usual 3 space are cusped edges and swallow tails — the surfaces, locally diffeomorphic to the set of polynomials $x^4 + ax^2 + bx + c$, having multiple (real) roots (this swallowtail hypersurface has been studied by Kronecker).

Example 6. Let us consider a smooth hypersurface in a Euclidean space (the origin excluded). The family of all the hyperplanes, normal to the radius-vectors of the hypersurface of all the points of the hypersurface has an envelope. This envelope may have singularities. Some experimentation shows that they are the same as those of the fronts, equidistants of the graphs of the Legendre transformations and so on, as in the preceeding example.

The geometric reason for these coincidences is that in all these different cases, there is somewhere a Legendre singularity.

THEOREM. *The transformations leading to the singular hypersurfaces of examples 5 and 6 may be described as products of the projective duality transformation and the inversion transformation.* (The order of the use of these two involutions in the two examples is different.)

Since the inversion transformation is a diffeomorphism, the singularities of examples 5 and 6 are Legendre singularities.

Example 7. Let $V = V^{2n}$ be a vector space of dimension $2n$, equipped with a linear symplectic structure (a nondegenerate bilinear skew-symmetrical form). The projectivized space P^{2n-1} carries a natural contact structure. (We associate to a point of V its skew-orthocomplement hyperplane, thus defining a hyperplane field on P^{2n-1}.)

This contact structure is invariant under the natural action of the linear symplectic group on P^{2n-1}.

Example 8. Let $V = \{a_0x^d + \ldots + a_dy^d\}$ be the vector space of all binary forms of degree d. The group SL_2 of linear unimodular transformations of the (x, y)-plane acts on V. If d is odd, the dimension of V which is $d+1$ is even and V carries a linear symplectic structure, invariant under the SL_2 action. This 2-form is unique up to a nonzero multiple. It defines on V a translation invariant symplectic structure.

[The explicit formula for this structure may be described in terms of the Darboux coordinates reducing the symplectic form to $\Sigma dp_i \wedge dq_i$. The expression of the binary form $\phi \in V$ in terms of these Darboux coordinates is

$$\phi(x, 1) = q_1 X_d + \dots + q_n X_n - p_n X_{n-1} + \dots + (-1)^n p_1$$
$$\text{where} \quad X_j = x^j/j!\,, \quad d = 2n-1$$

and the signs preceeding the p_m's alternate.]

Combining this natural symplectic structure of the space of binary forms with the construction of example 7 above we obtain:

PROPOSITION. *The projective space of the 0-dimensional hypersurfaces of degree $d = 2n-1$ of the projective line carries a natural $PL(2)$-invariant contact structure.*

[For instance, in the domain of the projective space of hypersurfaces represented by the polynomials $\phi(x, 1)$ with $q_1 = 1$, the above contact structure is defined as the set of zeroes of the 1-form

$$\alpha = p'dq' - q'dp' - dp_1$$
$$p' = (p_2, \dots, p_n), \quad q' = (q_2, \dots, q_n)] .$$

COROLLARY. *The set of 0-dimensional hypersurfaces of degree $2n-1$ containing any given point with multiplicity n (or higher than n) is a Legendre submanifold.*

This follows from the above explicit formula for the case of the point $x = 0$. Since the contact structure is PL_2 invariant, it is still true for any given point.

Example 8. The Gibbs (1873) "graphical methods in thermodynamics" is pure contact geometry. Let v be the volume, p the pressure, t the (absolute) temperature, ε the energy, η the entropy. The contact structure of thermodynamics is defined by the equation

$$d\varepsilon = t\,d\eta - p\,dv\,.$$

Every substance is represented as a Legendre 2-surface in this 5-dimensional contact manifold of the thermodynamical states. Different physical states of the same substance correspond to different points of this Legendre surface, but the evolution of state is severely restricted by the contact structure.

§2. CHARACTERISTICS

Let us consider a hypersurface in a contact manifold.

Example 1. A hypersurface in the manifold of 1-jets of functions is called a 1st order nonlinear differential equation.

Example 2. Let us consider pseudo-Euclidean space-time. Among all the contact elements of the space-time one distinguishes the light elements: those tangent to the light cone. The light elements form a hypersurface in the contact space of all the contact elements of the space-time.

More generally, a hyperbolic PDE (or a hyperbolic system of PDE) defines a field of "Fresnel cones" (of cones of zeroes of the principal symbol) in the cotangent spaces of the space-time manifold. This field defines the "light hypersurface" in the contact space of all the contact elements of the space-time manifold. The contact geometry of this hypersurface is crucial for the understanding of the propagation of the waves defined by the hyperbolic equation (or system).

The tangent plane to a hypersurface in a contact manifold is generically different from the contact plane (they coincide, generically, at some isolated singular points of the hypersurface). Let us first consider the nonsingular points. At a nonsingular point the hypersurface in the contact manifold carries a distinguished tangent line, called the characteristic direction. The word "characteristic" in mathematics always means "intrinsically associated". Thus the characteristic equation of a matrix of an operator is independent of the basis, the characteristic subgroups are invariant under automorphisms, and so on.

The *characteristic direction* at a given point of a hypersurface in a contact manifold may be defined as the only direction associated intrinsically to the hypersurface and to the contact structure. Indeed, the subgroup of the contactomorphisms which preserve the point and the hypersurface, preserves exactly one line tangent to the hypersurface at this point. (We still suppose that the tangent plane of the hypersurface is transversal to the hyperplane of the contact structure at this point.)

One might prefer a more explicit definition of the characteristic direction. For the simplest case of a 3-dimensional contact manifold it is the intersection of the contact plane with the tangent plane of the hypersurface at the given point.

In the general case the intersection is not a line, but a hyperplane in the contact hyperplane. To transform a hyperplane into a line, we use the natural (conformally) symplectic structure of the contact hyperplane.

Let α be a contact form, defining (locally) our contact structure, $\alpha = 0$. The 2-form $d\alpha$ defines the bilinear symplectic structures in the contact hyperplanes $\alpha = 0$. $d\alpha$ is nondegenerate at the plane $\alpha = 0$, since α is a contact form. The 1-form α is defined up to multiplication by nonzero functions. Since the 2-form $d(f\alpha) = df \wedge \alpha + fd\alpha$ is reduced to $fd\alpha$ at the plane $\alpha = 0$, the bilinear symplectic form $d\alpha$ in the plane $\alpha = 0$ is well defined up to a nonzero constant multiple.

The characteristic direction at a point of a hypersurface in a contact manifold is the skew-orthocomplement to the intersection of the tangent plane of the hypersurface with the contact plane at the given point. The characteristic direction of a hypersurface is tangent to the hypersurface, since the skew-orthocomplement of a hyperplane in a symplectic space belongs to the hyperplane.

The integral lines of the field of characteristic directions are called the characteristics of the given hypersurface.

Example. Let us consider the hypersurface $K = 0$ in the space with contact structure $\alpha = 0$, where

$$\alpha = dz + \frac{pdq - qdp}{2}.$$

[After some calculations] one finds the equation of the characteristics:

$$\dot{p} = -K_q + \frac{p}{2}K_z, \quad \dot{q} = K_p + \frac{q}{2}K_z, \quad \dot{z} = -\frac{p}{2}K_p - \frac{q}{2}K_q.$$

Example. The characteristics of the light hypersurface in the projectivized cotangent bundle of the space-time manifold defines the wave propagation for the corresponding (pseudo) differential hyperbolic equation. This is one more form of the Huygens principle.

A submanifold of a contact manifold, tangent to the contact planes, is called an *integral submanifold* of the contact structure. For instance, a Legendre submanifold is a maximal integral manifold.

Let us consider an integral submanifold of the contact structure, lying in a given hypersurface of the ambient contact manifold. Let us suppose that the characteristics of the hypersurface are nowhere tangent to the submanifold. An easy fact on characteristics is the following

THEOREM. *The characteristics, issuing from points of the given integral submanifold, form (at least locally) an integral submanifold.*

COROLLARY 1. *The characteristics of a Legendre submanifold belong to the submanifold.*

This property of the characteristics can be used as their definition.

COROLLARY 2. *If the dimension of the integral submanifold in the theorem is one less than the dimension of a Legendre manifold, then the characteristics intersecting the submanifold form (at least locally) a Legendre submanifold — the unique Legendre submanifold of the hypersurface containing the given integral submanifold.*

This corollary contains the theory of the Cauchy problem for first order nonlinear PDEs. Such a PDE $F(p, q, y) = 0$ defines a hypersurface in the space of 1-jets of functions $y(q)$, equipped with its natural contact structure $dy = pdq$. The initial data define an integral submanifold. The characteristic's equation is

$$\dot{q} = F_p, \quad \dot{p} = -F_q - pF_y, \quad \dot{y} = pF_p.$$

The Legendre submanifold, formed by the characteristics, is the 1-graph of the solution.

As another example let us consider a hypersurface in space. The contact elements of space-time, tangent to this submanifold and belonging to the light hypersurface, form an integral submanifold of the contact structure. The characteristics of the light hypersurface, issuing from points of this submanifold, form a Legendre submanifold of the light hypersurface in the space of contact elements of the space-time which is called the *big front*. The intersections of the big front with the isochrones are called the *momentary fronts*.

Thus the light hypersurface of the projectivized cotangent bundle of the space-time (considered as a hypersurface in a contact manifold) defines both the evolution of the wave fronts (the Legendre manifold) and the rays (the characteristics). This description of the wave propagation is one more formulation of Huygens' principle.

There exist two other types of characteristic in a contact manifold. Let us consider a contact form α on a contact manifold (a form, defining its contact structure, $\alpha = 0$). Such a form might be nonexistent globally and it is not intrinsically defined by the contact structure, but we can choose it locally.

The 2-form $d\alpha$ is as nondegenerate as possible for a skew-symmetric 2-form in an odd-dimensional space — it has at every point a 1-dimensional kernel. This kernel direction is called the *characteristic direction of the contact 1-form* α. It is the only direction, associated to the 1-form intrinsically (i.e. preserved by the contactomorphisms which do not change the form).

The integral lines of this field of directions are called the characteristics of the 1-form. They are transversal to the contact planes.

Example 1. The characteristics of the form $dz - pdq$ are the vertical lines generated by the vector field $\partial/\partial z$. The same is true for the form $dz + \dfrac{pdq - qdp}{2}$.

Example 2. Let us consider the standard embedding of S^3 in $\mathbf{C}^2$ or in the standard symplectic $\mathbf{R}^4$. The complex structure of $\mathbf{C}^2$ defines on S^3 a field of complex lines, which is a contact structure.

The symplectic structure of $\mathbf{R}^4$ also defines on S^3 a field of 2-planes (consider the skew-orthocomplements to the radius-vectors). This 2-plane field coincides with the contact structure induced on S^3 by the complex structure of $\mathbf{C}^2$ and is called the *natural* (or *standard*) *contact structure* of S^3.

This structure may be defined globally as the field of zeroes of an $SU(2)$-invariant 1-form. This 1-form is unique up to a scalar multiple.

The characteristics of this 1-form are some of the great circles of the sphere. They are the fibres of the Hopf fibration $S^3 \to S^2$.

In both these examples the characteristics of a contact form form a symplectic manifold. Indeed, the 2-form $d\alpha$ is well defined on the space of characteristics, since it vanishes along the characteristics.

COROLLARY. *Let us consider any Legendre curve (that is, any integral curve) of the contact structure in* $\mathbf{R}^3$ *defined by the contact form* $dz - pdq$.

Then there exists at least one characteristic chord of this curve (that is, at least one characteristic of the form intersecting the curve twice).

Proof. Let us consider the projection of the curve along the characteristics to the space of the characteristics (that is the projection to the (p, q-plane) along the vertical z-lines). Since the curve was a Legendre curve, its projection is a 0-area curve ($\oint pdq = 0$). Hence the projection has self-intersections, and the Legendre curve has characteristic chords.

An old conjecture on the contact 1-forms defining the standard contact structure on S^3, is the existence of characteristic chords for any Legendre curve (warning: a characteristic chord may have geometrically only one common point with the Legendre curve if this chord is closed, i.e. diffeomorphic to a circle).

This is a particular case of a set of general conjectures in the higher-dimensional contact topology of Legendre manifolds of greater dimension. These conjectures are not discussed here since even the simplest case of Legendre curves in S^3 still remains unsettled.

The two-sided relation between the contact and symplectic geometries depends on the fact that one can obtain an even number from an odd one either by adding or by subtracting one.

THEOREM. *The manifold of the characteristics of a contact form has a natural symplectic structure.*

Example. Let us consider the unit sphere $S^{2n-1} \subset \mathbf{C}^n$ with its natural contact form. The space of characteristics is $\mathbf{CP}^n$ with its natural symplectic form. Let us consider a contact structure, defined globally by a contact form.

THEOREM. *The total space of the bundle of all the linear forms on the tangent spaces to a contact manifold, which are zero exactly on the contact hyperplanes, has a natural symplectic structure.*

Indeed, this manifold is fibred over the contact manifold (with fibres $\mathbf{R}\backslash 0$). Each point of the bundle is a linear form on the tangent space to the total space at our point. This way we define a canonical 1-form α on the total space. The required symplectic structure is $d\alpha$.

The symplectic manifold thus obtained is called the symplectization of the original contact manifold.

Conversely, one may start from a symplectic manifold and obtain (at least locally) a contact manifold whose dimension is greater by 1 or smaller by 1 than that of the symplectic manifold.

The two standard models for this are the fibrations

$$\begin{array}{ccc} J^1(V, \mathbf{R}) & \to & T^*V \\ \text{contact} & & \text{symplectic} \\ dy - pdq & & dp \wedge dq \end{array}$$

and

$$\begin{array}{ccc} T^*B\backslash B & \to & \mathbf{P}T^*B \\ \text{symplectic} & & \text{contact} \\ dp \wedge dq & & pdq = 0\,. \end{array}$$

We may think of $J^1(V, \mathbf{R})$ as being the *contactization* of T^*V and of $T^*B\backslash B$ as being the *symplectization* of PT^*B.

Example. Let L be an (immersed) Legendre submanifold of $J^1(V, \mathbf{R})$. Then its projection to T^*V is an (immersed) Lagrange submanifold of T^*V.

Let us start with an arbitrary connected Lagrange submanifold Λ in T^*V. Let us fix a point in $J^1(V, \mathbf{R})$ over some point of Λ. Then locally there exists one and only one Legendre manifold in $J^1(V, \mathbf{R})$ containing the fixed point and projecting diffeomorphically to Λ ($y = \oint pdq$ is locally independent of the path joining two given points).

But globally such a Legendre manifold might exist or not exist. In the first case the initial Lagrange manifold Λ is called an *exact Lagrange manifold* (since in this case the 1-form pdq is exact on Λ).

For instance, the graph of the differential of a function on V is an exact Lagrange submanifold of T^*V.

Definition. A quasifunction on V is an imbedded Legendre submanifold of $J^1(V, \mathbf{R})$, which is isotopic to the zero section in the class of the imbedded Legendre submanifolds.

A quasicritical point of a quasifunction is a point where its Lagrange projection intersects the zero section of T^*V.

THEOREM (Tchekanov). *The number of quasicritical points of a quasi function on a compact manifold is bounded below by the sum of the Betti* *numbers of the manifold (when counted with multiplicities) and the number of geometrically different critical points is bounded below by the cuplength of the manifold.*

One conjectures that both the "algebraic" and the "geometric" numbers are bounded below by the minimal "algebraic" and "geometric" numbers of critical points of a function on the manifold. But this is proved not for the functions on the original manifold, but for functions on a vector bundle over the manifold coinciding with a nondegenerate quadratic form of signature zero at the infinity of every fibre.

The characteristics of the third type are the orbits of the flows of contactomorphisms. Let us suppose that the contact structure is defined by a global 1-form α and let us fix this 1-form.

A vector field on a contact manifold is called a *contact vector field,* if its flow preserves the contact structure. The contact vector fields form a Lie algebra — the Lie algebra of contact vector fields.

Let V be a contact vector field and let α be the fixed contact form. Then the function $K = \alpha | V$ is well defined. This function is called the *contact Hamilton function.* If we choose another contact form the contact Hamilton function acquires a nowhere zero functional multiplier. Hence the hypersurface (the "divisor") of the zeroes of K is intrinsically defined by the contact structure and by the contact vector field. The choice of a global contact form is a trivialization of the line bundle of the linear forms vanishing on the contact planes. The contact Hamilton function K is in fact a section of the dual line bundle. But I prefer to fix a trivialization and to call K a function.

THEOREM. *Any function on a contact manifold is the contact Hamilton function of some contact vector field, which is defined uniquely by this function.*

In Darboux coordinates, where

$$\alpha = dz + \frac{pdq - qdp}{2}$$

the contact vector field V with the contact Hamilton function K defines the "contact Hamilton differential equations"

$$\dot{p} = -K_q + \frac{p}{2}, \quad \dot{q} = K_p + \frac{q}{2} K_z, \quad \dot{z} = K_p + \frac{q}{2} K_z .$$

Since the last part of the theory is computational, it might be easier to understand it from the symplectic geometry point of view.

Let us consider the manifold of 1-forms in the tangent spaces to our contact manifold which vanish exactly on the contact planes.

This manifold has a natural symplectic structure (defined above) and a natural action of the multiplicative group $\mathbf{R}^*$ of nonzero scalars. It is a principal $\mathbf{R}^*$ bundle over the initial contact manifold.

We lift the contactomorphisms to the bundle space and we obtain the symplectomorphisms, commuting with the action of the multiplicative group. The corresponding Hamilton functions may be chosen to be the homogeneous functions of degree 1 along the fibres.

The choice of the bundle's trivialization α associates to these homogeneous functions contact Hamilton functions. The formula for the contact vector field is the usual formula for the Hamilton field, projected to the contact base from the symplectic total space (taking into account the homogeneity).

In the same way, the usual Poisson bracket on the symplectic $\mathbf{R}^*$ bundle space, restricted to the homogeneous functions of degree 1 along the fibres, defines a "Lie bracket" on the space of functions on a contact manifold. This contact Lie bracket is a particular case of a "Lie structure" on a manifold: it is a Lie algebra structure on the space of functions, the bracket being a first order differential operator in each of its two arguments.

Lie structures are generalizations of *Poisson structures* (which correspond to brackets homogeneous in the derivatives). A Poisson manifold is decomposed into a collection of *symplectic leaves* (manifolds of different dimensions) unified by the smoothness of the common Poisson bracket. The simplest example is the Lie-Berezin-Kostant-Kirillov bracket on the coadjoint space of a Lie algebra. For the $SO(3)$ case the leaves are defined by the equation $M_2 = \text{const}$ (they are spheres of dimension 2 when the constant is positive, and their common centre — a symplectic manifold of dimension 0 — when the constant vanishes).

The leaves are defined as the sets of points, attainable from a given one by Hamiltonian paths, [namely a Poisson structure on a manifold associates to a function on the manifold a vector field such that the Poisson bracket with this function is differentiation along this field. This field is called the *Hamiltonian field,* associated to a given Hamilton function. The Hamiltonian paths are defined by the time-dependent Hamilton functions (or, equivalently, by broken paths containing several segments of the time-independent Hamiltonian field's orbits).]

For Poisson structures there exists an elaborate theory of singularities, mainly reducing them locally to the transversal slice of the singular leaf

(A. Weinstein). In coordinates the Poisson structure has the form $\{a, b\} = \Sigma c^i_{jk}(x)\partial a/\partial x_j \partial b/\partial x_k$. For the transversal structure all the coefficients c vanish at the origin.

The linear part of the transversal structure defines a finite dimensional Lie algebra. If this algebra is semisimple, the whole transversal structure is equivalent to its linear part ([Co]). To be precise, the last theorem holds for analytic Poisson structures. In the C^∞ case the semisimple algebra should be the Lie algebra of a compact Lie group.

The problem of the linearization of transverse Poisson structures was discussed in my 1963 paper [A1]. The reduction of the 3-body problem here is based on the study of the Poisson leaves for $O(3)$.

Interesting Poisson structures are provided by the period mappings of the versal deformations of singular hypersurfaces ([GV]). The simplest of these Poisson structures lives in the three-dimensional (a, b, c) space of the swallowtail. [It is described axiomatically by the 3 conditions: (i) all the leaves are two-dimensional, (ii) the self-intersection line of the swallowtail is contained in one of the leaves, and (iii) the leaf containing the origin is transversal to the tangent plane of the swallowtail at the origin. For the swallowtail $\{x^4 + ax^2 + bx + c = (x+t)^2 \ldots\}$ the tangent plane is $dc = 0$.]

One can locally reduce any such structure to the normal form $\{a, c\} = 1$, $\{a, b\} = \{b, c\} = 0$ by a local swallowtail-preserving diffeomorphism. The leaves are the vertical planes $b = \text{const}$. The Poisson structure's normal form implies this normal form of the corresponding fibration of the space into planes.

A Lie structure on a manifold, like a Poisson structure, defines a decomposition of the manifold into submanifold leaves. In this case some of the leaves are even dimensional symplectic manifolds, while others are odd dimensional contact manifolds ([Ki]).

The transversal structure theory for this case has not been yet constructed, as far as I know.

From the algebraic point of view Poisson structures are better objects than honest symplectic manifolds, and general Lie structures are better than nondegenerate contact ones.

The algebraic objects include automatically all the degenerations of the corresponding geometric structures. Perhaps because of this advantage the algebraic theory is so difficult that it contains almost no general results. The few general results that I know were first obtained geometrically (for some mild degenerations) and then the geometrical proofs were translated into the algebraic language, and hence had become general. The dictionary of the translation is simple: one substitutes principal ideals for the hypersurfaces, one controls the Poisson brackets vanishing at subvarieties by corresponding conditions on the ideals and so on. For instance, the nondegeneracy condition for the hypersurface $f = 0$ is "$\{f, g\} = 1$ for some function g". The algebraic study of the degenerate symplectic and contact varieties is an important but almost unexplored domain.

For instance, let us consider the generalized swallowtail surface $\{x^5 + Ax^3 + Bx^2 + Cx + D = (x+t)^2 \ldots\}$ in the 4-space of the versal deformation of the singularity A_4 (i.e. x^5 or $SU(5)$). The Givental-Varchenko period mapping equips this 4-space with a natural symplectic structure (one may find an explicit formula in the last chapter of the book [AGV]). The generalized swallowtail has with respect to this structure very special properties (for instance, the selfintersection subvariety $\{(x+t)^2(x+s)^2 \ldots\}$ is Lagrangian, because different cycles, vanishing at the same critical level of a function, do not intersect each other. But it is unkown whether this swallowtail is uniquely defined (up to symplectomorphisms) by the ranks of the restrictions of the symplectic form to the tangent cones of its strata.

Let us return to a usual contact space, equipped with its Darboux coordinates (p, q, z) and with its Darboux contact form $\alpha = dz + \dfrac{pdq - qdp}{2}$.

Let K, as above, be the contact Hamilton function of a contact vector field V. We shall denote by a dot over the name of a function the derivative of this function along V. The explicit formula for the components of the field V in the Darboux coordinates is given above (it precedes the Lie structure discussion). This formula implies the

COROLLARY. $\dot{K} = KK_z$, $L_V\alpha = K_z\alpha$ *(where* L *is the Lie derivative). If* $H(p, q)$ *is a quadratic form, then*

$$\dot{H} = \{K, H\} + HK_z$$

where $\{K, H\} = K_pH_q - H_pK_q$ *is the symplectic Poisson bracket.*

Now we shall compare the three types of characteristics.

Let us fix a contact 1-form α and a function K. Let us consider all the three characteristic directions at every point: that of the hypersurface $K = \text{const}$, that of the 1-form α and that of the contact Hamilton vector field, defined by K.

In general they are different.

THEOREM. *These three lines lie in a 2-plane.*

Proof. This is clear from the above formula for the vector field V in Darboux coordinates and from the formula for the characteristics of a hypersurface. The only difference is the additional term $K\partial/\partial z$ in the formula for V. This additional term is vertical, i.e. directed along the characteristics of the 1-form α.

Exercise. Find a direct proof, independent of the Darboux coordinates.

The tangent hyperplanes of a generic hypersurface in a contact manifold coincide with the contact planes at some isolated points of the hypersurface. Those points are called the characteristic points. They are the singular points of the field of the characteristic directions.

A smooth generic hypersurface may be reduced to a simple normal form in a neighbourhood of its characteristic points by a smooth contactomorphism (Lychagin, 1975). In Darboux coordinates the Lychagin normal form is

$$z = Q(p, q),$$

where Q is a nondegenerate quadratic form. In the complex case one may reduce Q further to the "eigenvalues" normal form $Q = \Sigma\lambda_k p_k q_k$ (since any linear symplectomorphism, $(p, q) \mapsto (\tilde{p}, \tilde{q})$ induces a contactomorphism $(p, q, z) \mapsto (\tilde{p}, \tilde{q}, z)$).

The classification of characteristic points of holomorphic hypersurfaces in contact manifolds is very different from this simple normal form. The formal series, which reduces the hypersurface and the contact structure to their normal forms is generically divergent.

However in the case of a 3-dimensional contact manifold the situation is better. Let us consider an implicit ordinary differential equation defined by the hypersurface $F(p, q, y) = 0$ in the 3-space of 1-jets of functions $y = f(q)$ equipped with its natural contact structure $dy = pdq$ and with the structure of the Legendre fibre bundle $J^1 \to J^0((p, q, y) \mapsto (q, y))$.

The characteristics of the surface $F = 0$ projected to the base (q, y)-plane, are called "integral curves" of the implicit differential equation.

For a generic function F the surface $F = 0$ is smooth, but its projection to the base plane may have singularities. Those singularities are generically fold lines and ordinary Whitney cusp points.

The contact planes are vertical (they contain the fibre directions). Hence the characteristic points of the surface belong to the fold line (generically those points will not coincide with the cusp points).

THEOREM (A. Davydov, 1985). *In a neighbourhood of a characteristic point of the surface* $F = 0$ *a generic implicit differential equation may be reduced to the local normal form*

$$(*) \qquad y = (p+kq)^2$$

by a local diffeomorphism of the base (y, q)*-plane.*

This diffeomorphism is C^∞ for the C^∞-equations, analytic for the analytic equations and so on. The (real) number k is a modulus — a parameter, on which the normal form depends.

A (local) diffeomorphism of the base space of our Legendre fibration induces a (local) contactomorphism, of the fibration total space. Hence the surface, $F = 0$ is reduced to the special form (*) by a (local) contactomorphism. Thus the family of characteristics on a generic surface in a contact 3-space is locally diffeomorphic (in a neighbourhood of a characteristic point) to the family of the characteristics on the standard surface (*).

The differential equation (*) is easy to solve and the corresponding phase curves on the surface form (for any generic value of k) one of the standard Poincaré patterns: the node, the focus, the saddle.

The folding mapping defines an involution on the surface (interchanging the two counterimages of any base point). The essential part of Davydov's proof is the proof of the equivalence of all the generic involutions, occurring in such a situation. Namely one considers the involutions of the plane, whose curves of fixed points contain the singular point of a vector field, this field being anti-invariant under the involution at the line of fixed points of the involution. Those two properties of the involution and the genericity are sufficient for the stability of the involution with respect to a generic vector field: all the neighbouring involutions with the same properties may be transformed into a given one by a diffeomorphism which preserves orbits of the vector field.

The study of singularities of implicit equations was one of the 4 problems in King Oscar II of Sweden's list of 1985. (One of the other problems was the 3-body problem and the prize-winning paper of Poincaré was his memoire on the 3-body problem).

The topological structure of the singular points of implicit ordinary differential equations was settled by A.V. Phakadze and A.H. Shestakov (1959); later this subject was studied independently by Thom (1972), L. Dara (1975), F. Takens (1976). These last three mathematicians have conjectured that (*) is a topological normal form (this fact was also proved by Phakadze and Shestakov and then rediscovered by Piliy and Fedorov, 1970, in the context of plasma physics).

The Davydov theorem implies that (*) is even a smooth and analytic normal form, which goes much further than all the preceding conjectures in this domain.

It is interesting to note that these mathematical results of contact geometry have many different interpretations in terms of different branches of mathematics and physics.

For instance, the same normal form describes the patterns of asymptotic lines at some points of a parabolic line on a generic surface in Euclidean 3-space.

It provides also the normal forms for the field of characteristics of a generic partial differential equation of mixed type, (whose type changes from elliptic to hyperbolic along some line in the plane of the independent variables).

One may also describe the same Davydov theorem in terms of the theory of relaxation oscillations in systems with two slow and one fast variables. In this case the contact structure is defined by a vertical (fast) vector field and by a small (slow) perturbing generic vector field in the 3-space of slow and fast variables. The Legendre fibration is defined by the (vertical) direction field of the fast motion. The surface in the contact space is formed by the equilibria points of the fast motion. It is usually called the slow surface in relaxation oscillation theory. The characteristics of this surface describe the orbits of the slow motion (occurring after the relaxation of the fast motion).

One more example of the application of the Davydov theorem is provided by the theory of Newton's equation with 1 degree of freedom $\ddot{x} = F(x) - k\dot{x}$.

Let us consider the (x, E) coordinate energy plane, $E = \dot{x}^2/2 + U(x)$, $F = -\partial U/\partial x$.

The projection of the motion to the (x, E)-plane defines a family of curves in the domain $E \geqslant U(x)$. At the points of the potential wells and of the barriers (of the minima and of the maxima of U) these curve families have singularities. The type of the singularity depends on the value of the friction coefficient k and on the type of the critical point of the energy. But the family is generically diffeomorphic to the family of integral curves in the (q, y) plane of Davydov's theorem.

Thus the general theorems of contact geometry unify many apparently different theories in many branches of mathematics and of physics, making transparent the common mathematical structures and features of all these theories. Hence the contact geometry strategy is to translate the intuition and concrete results of any of the branches of its application to all the other branches.

Let us consider a hypersurface in a contact manifold, which has no characteristic points. Such a hypersurface is foliated (locally fibrated) into characteristics. The set of characteristics is (at least locally) a manifold whose dimension is two less than the dimension of the initial contact manifold.

THEOREM. *The manifold of characteristics inherits a contact structure from the initial contact manifold.*

A formal way of proving this theorem is a direct calculation. According to the preceding formulae, the characteristics of the hypersurface $K = 0$ are the orbits of the vector field $W = V - K\partial/\partial z$, where $L_V\alpha = K_z\alpha$. Since $i_{\partial/\partial z}d\alpha = 0$, $i_{\partial/\partial z}\alpha = 1$, we have

$$L_W\alpha = K_z\alpha - dK\,.$$

The second term vanishes along the hypersurface $K = 0$. Thus the flow of the vector field W on the hypersurface $K = 0$ preserves the field of hyperplanes $\alpha = 0$ and hence defines a field of hyperplanes on the space of orbits of this field.

Another way of proving this theorem is to consider just one particular case, say the hypersurface, defined in Darboux coordinates by the equation $p_1 = 0$. In this case the characteristic direction is $\partial/\partial q_1$. Hence the space of the characteristics is the coordinate space with Darboux coordinates (z, p, q) where $\tilde{p} = (p_2, \ldots, q_n)$ $\tilde{q} = (q_2, \ldots, q_n)$. Thus the form $\alpha = dz + \dfrac{pdq + qdp}{2}$ induces on the manifold of characteristics of the hypersurface $p_1 = 0$ the form $\tilde{\alpha} = dz + \dfrac{\tilde{p}d\tilde{q} - \tilde{q}d\tilde{p}}{2}$, as was required.

Now the general case can be reduced to this particular case, since all the hypersurfaces in a contact manifold are locally contactomorphic in neighbourhoods of their non-characteristic points, which follows from the general theorem of Givental, described below.

§3. SUBMANIFOLDS

The submanifolds of a Euclidean or a Riemannian manifold have interior and exterior geometries. For instance, the Gaussian curvature belongs to the interior geometry of the Riemannian metric on the submanifold, while the mean curvature depends on its exterior geometry. In both symplectic and contact geometries the situation is simpler: the local exterior geometry is reduced to the interior one.

THEOREM (Givental). *A submanifold of a contact manifold is locally defined (up to a contactomorphism) by the restriction of the contact structure to the submanifold.*

The contact structure is here considered locally as the module of differential 1-forms vanishing at the contact hyperplanes (over the algebra of functions). The restriction is the module of the restrictions of these forms to the submanifold (over the algebra of functions on the submanifold). Geometrically this means that we consider the field of intersections of the contact planes with the tangent planes of the submanifold, taking the multiplicities into account.

The above theorem follows from a similar theorem on contact forms. We call a contact form transversal to a submanifold if the characteristic directions of the form are nowhere tangent to the submanifolds.

THEOREM (Givental). *Let us consider a homotopy α_t of 1-forms defining (different) contact structures on a manifold. Suppose that they are all transversal to a given submanifold, and that they coincide on all the tangent vectors to the submanifold. Then there exists a diffeomorphism of a neighbourhood of the submanifold, identical on the submanifold and transforming α_0 to α_1.*

Proof. By a standard homotopy argument the problem is reduced to the solution of the "homology equation"

$$L_{V_t}\alpha_t = \beta_t, \qquad \beta_t = -\partial\alpha_t/\partial t,$$

where V_t is the unknown vectorfield, vanishing on our submanifold M and β_t is a known 1-form, vanishing on the tangent vectors to M.

Since the smooth dependence of V on t is easily attainable in the construction that follows, we omit from now on the subscript t to simplify the notations.

Let $V = U + W$ be the decomposition of the (unknown) vector field into its horizontal and vertical parts:

$$i_U\alpha = 0, \qquad i_W d\alpha = 0.$$

Let W_0 be a vertical vector for which $i_{W_0}\alpha = 1$. Then $W = fW_0$, $f = i_W\alpha$. By the homotopy formula $L_V = i_V d + d i_V$ we obtain

$$L_V\alpha = i_U d\alpha + df.$$

Thus the homology equation has the form

$$i_U d\alpha + df = \beta$$

where β vanishes at the tangent vectors of M and both the unknown function f and the unknown horizontal field U should vanish on M.

Let π be the geodesic projection of the tubular neighbourhood of M onto M and p be the operator, associating to any k-chain c in this neighbourhood the $k+1$ chain pc formed by the shortest paths of its points x to πx. Then $\partial(pc) + p(\partial c) = c - \pi c$. Let $p^! := \Omega^k \to \Omega^{k-1}$ be the dual of p on the differential forms defined by the identity $\int_{pc} \omega = \int_c p^! \omega$. It is clear that $p^!\omega$ is a differential form, vanishing at the points of M on every k-vector of the ambient space. By duality

$$p^! d\omega + dp^!\omega = \omega - \pi^*\omega \ .$$

For $\omega = \beta$, $\pi^*\beta = 0$ since β vanishes on the tangent vectors of M. Hence $\beta = p^! d\beta + dg$, $g = p^!\beta$. The function g vanishes on M and the form $p^! d\beta$ vanishes at the points of M on every vector of the ambient space. Now the homological equation takes the form

$$i_U d\alpha + df = p^! d\beta + dg \ ,$$

where the unknown f and U should vanish at M. Evaluating the left and the right hand sides at the vertical vector W_0, we find

$$i_{W_0} df = h$$

where $h = i_{W_0} p^! d\beta + \dot{L}_{W_0} dg$ is a known function. This equation is easily solved for f (one integrates along the verticals). The solution f can be chosen to vanish on M (even on any hypersurface, containing M and transversal to the verticals, i.e. to the characteristics of the form α). After choosing f we obtain for U the residual homological equation

$$i_U d\alpha = \gamma \,,$$

where the 1-form $\gamma = p^! d\beta + d(g-f)$ is zero at the vertical: $i_{W_0}\gamma = 0$, by the choice of f. The condition $i_{W_0}\gamma = 0$ implies the solvability of the equation $i_U d\alpha = \gamma$ for U. The horizontal solution $U(i_U\alpha = 0)$ is unique, since the form $d\alpha$ is nondegenerate at the hyperplane $\alpha = 0$. Hence the horizontal solution U is smooth.

Now we use the ambiguity of the choice of f to reduce U to zero at M. The function f is defined uniquely adding a function which is constant along the characteristics and which vanishes at M. At the points of M the value $i_{W_0}d(g-f)$ is zero (since $p^! d\beta$ vanishes on M). Hence $d(g-f)$ coincides at the points of M with the differential of a function k, constant along the verticals ($k=g-f$ at some hypersurface, transversal to the verticals and containing M).

By adding k to f we do not change the value of df at W_0 and of f at M. But we change γ: now $\gamma = p^! d\beta$ at some hypersurface, containing M and transversal to W_0. Hence $\gamma = 0$ on every vector of the ambient space at the points of M. Thus now $U = 0$ on M, and hence $V = 0$ on M, as required.

The theorem on the contact structures is an easy corollary of this theorem on contact forms.

COROLLARY 1. *The Darboux theorem.*

Proof. $M =$ (point).

COROLLARY 2. *All the Legendre manifolds of any given dimension are locally equal (contactomorphic).*

Proof. All zeroes are equal.

COROLLARY 3. *All hypersurfaces in a contact manifold are contactomorphic in some neighbourhoods of any of its non-characteristic points.*

Proof. The restrictions of the contact forms are equivalent, since they are induced from the natural contact form on the spaces of the characteristics of the hypersurfaces, and those contact forms are equivalent by the Darboux theorem.

Thus we obtain the normal forms for the maximally non-degenerate 1-forms in the even-dimensional spaces: $\alpha = dz - ydx$ for a space of dimension $2n+2$ with coordinates $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$, z, w.

The simplest degenerations of the differential 1-form on a manifold are classified by J. Martinet [Ma]:

$$(1+p_1)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n)$$

$$\pm\, dz^2 + (1+p_1)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n+1)$$

$$(1 \pm p_1^2)dq_1 + p_2dq_2 + \dots + p_ndq_n \quad (\dim = 2n)\,.$$

Comparing with the Givental theorem, we obtain the

COROLLARY 1. *For a generic even dimensional submanifold of a contact space the nongeneric points form a set of codimension* 2, *and in a neighbourhood of the generic points the contact structure is maximally nondegenerate (reducible to the form,* $dz = ydx$ *for some coordinates* $x, y, z, w, \dim\{x\} = 1$*).*

COROLLARY 2. *A generic odd dimensional submanifold in a contact manifold inherits a contact structure at its generic points. At the points of some hypersurface the restriction of the contact structure to the manifold is reducible to one of the two (nonequivalent) forms* $\pm dz^2 = (1+p_1)dq_1 + p_2dq_2 + \dots + p_k dq_k$.

Remark. The above classification of the submanifolds depends on the classification of the contact structures (= modules of forms) and not on the forms' classification.

A differential 1-form in a neighbourhood of its nonzero point is either locally equivalent to one of the Darboux or Martinet normal forms, discussed above, or this form is not finitely-determined (is not determined by any finite segment of its Taylor series up to a diffeomorphism). The simplest example of such a nonfinitely determined form is the form $(1+y^3+xy)dy$ on the plane. The codimension of the corresponding event is two.

At present the classification of the degeneration of contact structures (not forms) has acquired the same level of sophistication as the other problems in singularity theory. In the works of M. Zhitomirskii the list of first degenerations is calculated, including all the simple singularities (a singularity is simple, if it has a neighbourhood intersecting a finite number of classes of equivalence). These results of Zhitomirskii, taking into account the Givental theorem, describe also the submanifolds in the contact space up to a local diffeomorphism.

Unfortunately, in most applications one needs the classification of nonsmooth subvarieties of the contact space, for instance, that of unions of intersecting submanifolds.

Example. Let us consider a hypersurface in a Riemannian space. The description of this situation in terms of the contact geometry implies the analysis of a pair of hypersurfaces in the contact space (the symplectic variant of this theory is developed by Sato, Oshiva and Melrose under the name of the theory of glancing rays).

Let us denote our closed Riemannian manifold by M, and the given hypersurface by ∂M.

We shall start from the contact manifold $J^1(M, \mathbf{R})$ of 1-jets of functions on M. Let us consider two hypersurfaces in this space:

$$SJ^1(M, \mathbf{R}) \to J^1(M, \mathbf{R}) \leftarrow \partial J^1(M, \mathbf{R})\,.$$

The left hypersurface is defined by the Hamilton-Jacobi equation $p^2 = 1$. It is the contact equivalent of the Riemannian metric. The right hypersurface is formed by the jets of functions on M at the points of ∂M. It is the contact equivalent of the hypersurface ∂M in M.

We shall see that a large part of the Riemannian geometry of the hypersurface ∂M in M may be formulated in terms of these two hypersurfaces in the contact space. Since the contact geometry of these two hypersurfaces is (more or less) independent of their origin, we can apply the knowledge of Riemannian geometry and even the intuition of Euclidean space to the general case of an arbitrary pair of hypersurfaces Y, Z in a contact space X. Let us first consider this general situation.

The hypersurfaces Y and Z intersect generically along a submanifold W of codimension two in X (we suppose that the intersection is transversal). So we obtain the diagram of inclusions

$$\begin{tikzcd}
& X^{2n+1} & \\
Y^{2n} \arrow[ur] & & Z^{2n} \arrow[ul] \\
& W^{2n-1} \arrow[ul] \arrow[ur] \arrow[uu, hook] &
\end{tikzcd}$$

We shall also suppose that the hypersurfaces Y and Z are not tangent to the contact planes (that condition is generically satisfied at a neighbourhood of W since the characteristic points of the hypersurfaces Y and Z are generically isolated).

Hence each of the two hypersurfaces is foliated into its characteristics. Locally (and sometimes globally) this foliation is a fibration, that is there exists a space of characteristics (the base of the fibration). Let us denote the fibrations into characteristics by $Y^{2n} \to U^{2n-1}$ and $Z^{2n} \to V^{2n-1}$ (strictly speaking, U and V are defined only for the germs of Y and Z at a point of W).

Let us consider the composite mappings

$$(\text{via } Y)U^{2n-1} \leftarrow W^{2n-1} \to V^{2n-1}(\text{via } Z)\,.$$

These two mappings of manifolds of equal dimensions may have singularities. Let us consider the sets of their singular points (points, where the Jacobian matrix's determinant vanishes).

Under some very mild restrictions the sets of critical points of both mappings coincide. Indeed, let us suppose, that tangent hyperplanes of the hypersurfaces Y and Z and the contact hyperplane $\alpha = 0$ form a generic triple of hyperplanes (at some point 0 of W).

LEMMA. *The characteristic of the hypersurface* Y *is tangent to* Z *at the point* 0 *if and only if the restriction of the contact form of* X^{2n+1} *to* W^{2n-1} *degenerates at* 0.

Proof. Let us denote the intersections of the tangent planes to $X, \ldots, W$ at 0 with the contact hyperplane $\alpha = 0$ by the corresponding lower case letters $x, \ldots, w$. Let ξ be the characteristic vector of Y at 0. If ξ is tangent to Z, it belongs to w. Since ξ is skew-orthogonal to y, ξ is skew-orthogonal to w. Hence $d\alpha$ degenerate at w, as required.

Let $d\alpha$ be degenerate at w. Since $\dim w = 2n - 2$ is even, $\dim \operatorname{Ker}(d\alpha | w)$ is at least 2. Let η be a vector, transversal to w in y. Then the equation $d\alpha(\xi, \eta) = 0$ has nontrivial solutions $\xi \in \operatorname{Ker}(d\alpha | w)$. These solutions ξ are skew-orthogonal to η and to w. Hence they are the characteristic vectors of Y at 0. Thus the characteristic vectors of Y at 0 are tangent to W (and hence to Z), as was required.

The lemma is thus proved. Since the condition on the restriction of α to W in the lemma is symmetrical with respect to Y and Z, the lemma implies

COROLLARY. *The characteristics of the hypersurface* Y *are tangent to* W *at the same points as the characteristics of the hypersurface* Z.

Hence the sets Σ of the critical points of our mappings of W to U and to V coincide. We have thus obtained the following hexagonal commutative diagram of mappings

$$\begin{tikzcd}
 & X^{2n+1} & \\
Y^{2n} \arrow[ur, hook] \arrow[dd] & & Z^{2n} \arrow[ul, hook'] \arrow[dd] \\
 & W^{2n-1} \arrow[uu, hook] \arrow[ul, hook] \arrow[ur, hook'] \arrow[dl] \arrow[dr] & \\
U^{2n-1} & & V^{2n-1} \\
 & \Sigma^{2n-2} \arrow[uu, hook] \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where X^{2n+1}, U^{2n-1} and V^{2n-1} are equipped with contact structures, and Σ is the set of degenerescence of the restriction of the contact structure of X to W and at the same time the set of critical points of both mappings of W to U and to V.

The dimensions of the kernels of the derivatives of these mappings can't exceed one, since they are the restrictions of the corank 1 projections $Y^{2n} \twoheadrightarrow U^{2n-1}$ and $Z^{2n} \twoheadrightarrow V^{2n-1}$. Hence for the generic hypersurfaces Y and Z the singularities of the mappings $W \to U$ and $W \to V$ are, up to diffeomorphisms, the standard Whitney singularities.

One may even choose the coordinates in Y and U (or in Z and V) in such a way, that the hypersurface W in Y will be defined locally by the equation

$$y^{k+1} + u_1 y^{k-1} + \dots + u_k = 0$$

and the projection $Y \twoheadrightarrow U$ — by the formula

$$(y; u_1, \dots, u_{2n-1}) \mapsto (u_1, \dots, u_{2n-1}) .$$

Example 1. $k = 1$, $n \geqslant 1$. The mapping $W^{2n-1} \to U^{2n-1}$ has a fold singularity at the surface Σ^{2n-2} where $u_1 = y^2$, $y = 0$.

The characteristics $u =$ const. of Y intersect W twice in the neighbourhood of Σ, defining on W an involution. The hypersurface $\Sigma^{2n-2} \subset W^{2n-1}$ is the set of fixed points of this involution.

Hence at the generic points of Σ two involutions $W \to W$ are defined: one interchanges the two points of intersection of W with the characteristics of Y, the other — with the characteristics of Z. Both these involutions have the same hypersurface Σ of fixed points.

Example 2. $k = 2$, $n \geqslant 2$. The mapping $W^{2n-1} \to U^{2n-1}$ has a smooth hypersurface Σ^{2n-2} of critical points, which are the fold points or the cusp singularities. The cusp singularities form a smooth hypersurface $\Sigma^{2n-3} \to \Sigma^{2n-2}$. The set of critical values (the projection of Σ^{2n-2} to U) is a hypersurface in U with a cuspidal edge (projection of Σ^{2n-3}).

Now let us see what is the meaning of all this "general nonsense" in concrete situations.

Example. Let us return to the case of a hypersurface $\partial M : f(q) = 0$ in a Euclidean space $M = \mathbf{R}^n$. In this case $X = J^1(M, \mathbf{R})$, $Y : p^2/2 - 1/2 = 0$ is the Hamilton-Jacobi equation, $Z : f(q) = 0$ defines the hypersurface. The hexagonal diagram takes the form

$$
\begin{tikzcd}
 & & J^1(M, \mathbf{R}) & & \\
SJ^1(M, \mathbf{R}) \arrow[urr, hook] \arrow[dd, two heads] & & & & \partial J^1(M, \mathbf{R}) \arrow[ull, hook'] \arrow[dd, two heads] \\
 & & W \arrow[uu, hook] \arrow[ull, hook'] \arrow[urr, hook] \arrow[dll, "I"'] \arrow[drr] & & \\
ST^*M & & & & J^1(\partial M, \mathbf{R}) \\
 & & SJ^1(\partial M, \mathbf{R}) \arrow[uu] \arrow[ull, "\exp"] \arrow[urr] & &
\end{tikzcd}
$$

Comments. The characteristics of the Hamilton-Jacobi hypersurface $SJ^1(M, \mathbf{R})$ are the orbits of the geodesic flow in the space ST^*M of the spherical cotangent bundle, equipped with a parameter (the "value" of the jet), increasing along the geodesic with a velocity equal to one.

Fixing the value of this parameter, say $t = 0$, we obtain a point of the characteristic, that is a (cotangent) vector of length one at some point of M, equipped with the 0 "value". Thus we identify the space of characteristics with the space of the spherical cotangent bundle ST^*M (this identification *depends* on the choice $t = 0$).

The projection I associates to a point of ∂M, together with a vector on M of length 1 at that point and a "value" t, the unit tangent vector (on the same line as the original vector) based at a point at a distance t (in the backward direction) from the original point.

A characteristic of $\partial J^1(M, \mathbf{R})$ consists of the 1-jets of all the extensions of a fixed function on ∂M to M at some fixed point of ∂M. The manifold of characteristics is naturally identified with the manifold $J^1(\partial M, \mathbf{R})$ of the 1-jets of functions on ∂M, equipped with its natural contact structure.

The projection II: $W \to J^1(\partial M, \mathbf{R})$ associates to a 1-jet of a function on M, having gradient of length one, the 1-jet of its restriction to ∂M. This projection has the fold singularities on the surface Σ, formed by the 1-jets of the functions on M, whose gradients are of lengths one and are tangential to ∂M. The projection II maps the hypersurface Σ diffeomorphically to the set of critical values of this projection. This set of critical values consists of the 1-jets of functions on ∂M with gradients of length one. Hence we may identify Σ with $SJ^1(\partial M, \mathbf{R})$.

The mapping $\exp: SJ^1(\partial M, \mathbf{R}) \to ST^*M$ associates to a 1-jet of a function on ∂M, whose gradient has length one and is tangent to ∂M, a vector of length one on the same straight line as the given vector, but based at the point at a distance t (in the backward direction).

The singularities of the mapping I represent the "inflections" of ∂M.

Example. Let $n = 2$, that is ∂M is a generic plane curve. The mapping I has a fold singularity at the point [of W, corresponding to the unit tangent vectors of ∂M] where the curvature of ∂M is nonzero, and a cusp singularity at the [points of W corresponding to the] inflection points.

Let $n = 3$, that is ∂M is a generic surface in $\mathbf{R}^3$. The mapping I has folds at the points of W corresponding to the generic unitary tangent vectors, cusps at the vectors of asymptotic directions, the swallowtail singularity at the biasymptotic vectors (where the order of tangency of the surface with the tangent line is 3, which is higher than for an ordinary asymptotic vector). The biasymptotical directions exist on a generic surface along a curve; at some special points of this curve there exist triasymptotic directions, the order of tangency is 4 and in Whitney normal form for the singularity of the mapping I we have $k = 4$.

Thus the geometry of a hypersurface in a Euclidean (or in a Riemannian) space, when translated into the microlocal language of contact geometry, leads to the problem of classification (up to contactomorphism) of hypersurfaces with special singularities: of the unions of two smooth and transversal hypersurfaces (Y and Z) in a contact manifold X.

The simplest case ($k = 1$) was studied by Melrose. The normal form of the pair in Darboux coordinates is

$$q_1 = 0, \quad q_1 = p_1^2 + p_2 .$$

This is a formal (or C^∞?) normal form of a generic pair of hypersurfaces in a contact space. In the analytical case the normallizing series are, as a rule, divergent. In the 3-dimensional contact space the normal form of the pair is ($z = q, p^2 = q$) [Me]).

For further results on normal forms in the contact geometry of tangencies see [A3], [La] and [A6].

The state of art in this domain is at present far from the final death of the subject: in most cases the results are know only at the formal level of power series which are usually divergent in the analytical case.

Example 1. Let us consider the product of a swallowtail surface with a Euclidean space, defined in an N-space with coordinates $A, B, \ldots$ by the equation: $\exists t: x^4 + Ax^2 + Bx + C = (x+t)^2 \ldots \forall x$. A generic symplectic structure in a neighbourhood of the origin is formally reducible to the normal form

$$dA \wedge dD + dC \wedge dB + dE \wedge dF + \ldots$$

by a swallowtail preserving diffeomorphic; a generic contact structure — to the Landis normal form

$$\alpha = dZ - DdA - CdB - EdF - \ldots .$$

These normal forms serve probably in the C^∞ case too, but this is not proved.

Example 2. Let us consider a quadratic cone surface in a $2n + 1$ contact space with coordinates $A, B, \ldots$ given by $A^2 + B^2 = C^2$.

The local reduction of such a surface to a normal form by a contactomorphism is important for the study of the transformations of waves, defined by linear hyperbolical systems, derived from variational principles (see [A8] and [A9]).

The formal normal forms of the hypersurfaces with conical singularities in the Darboux coordinates ($\alpha = dz + \dfrac{pdq - qdp}{2}$) are

$$p^2 \pm q^2 = z^2 + cz^3 \quad (n=1), \qquad p_1^2 \pm q_1^2 = q_2^2 \quad (n>1).$$

These normal forms describe an interior transformation of waves of one kind (say "longitudinal") into waves of other kinds (say, "transverse") in inhomogeneous anisotropic media. The corresponding effect in homogeneous media is the Hamilton conical refraction. In the nonhomogeneous case the geometry of rays is different.

Let us consider the case $n = 1$, that is wave propagations for space-time of dimension 2. The preceding normal form describes two families of characteristics in space-time tangent at one point.

The characteristics through this point are formed from the branches of two smooth (analytic) curves, tangent at that point but having different curvatures. Let those curves be 12 and 34, then the first family's singular characteristic is 14, and that of the second — 32.

The contact of the two characteristics at the origin produces some singular scattering of the family of characteristics of the first (second) type, which is smooth (analytic) outside the origin. Let us consider a nonsingular characteristic of the first family, starting at a small distance ε from the point 1 (where the distances of points 1-4 to the origin are of order 1). The endpoint of this characteristic, taken at the level of point 4, lies at some distance from the singular characteristic 14, namely, at a distance $a\varepsilon + b\varepsilon^2 ln\varepsilon + \dots$. The logarithmic term describes the scattering at the origin: in a regular family the distance would be $a\varepsilon + b\varepsilon^2 + \dots$.

In the 3-dimensional physical space (i.e. for space-time of dimension 4) generic wave fronts (travelling in inhomogeneous media and governed by a variational principle) acquire singular lines, connecting them with waves of different kinds and moving with the wave fronts.

It is interesting to note that the case $n = 1$ is more difficult than $n > 1$. The results are at present formal in both cases. They probably hold for the C^∞ problem both for $n = 1$ and $n > 1$. The divergence of the normalizing series in the analytical problem is proven in the case $n = 1$, while for $n > 1$ there exists still some hope that the series converges. The qualitative results, described above, are independent of the convergence of the series: we need only finite segments of the series.

§4. Legendre fibrations and singularities

The simplest examples of Legendre fibrations are the projectivized cotangent bundles

$$PT^*V^n \to V^n$$

and the "forgetting of derivatives" mappings

$$J^1(M, \mathbf{R}) \to J^0(M, \mathbf{R})$$

(in coordinates: $(p, q, y) \mapsto (q, y)$).

Definition. A *Legendre fibration* is a fibration of a contact manifold with Legendre fibres.

THEOREM. *All the Legendre fibrations of a given dimension are locally contactomorphic (locally = in a neighbourhood of any point of the total space).*

To prove this theorem it is sufficient to construct a local isomorphism of an arbitrary Legendre fibration with one of the preceding examples.

Let us project a contact hyperplane from its contact point in the total space of the fibration to a base point. The image is a contact element (a hyperplane in the tangent space to the base of the fibration) since the tangent plane to the Lagrangian fibre lies in the contact hyperplane.

Thus we have defined a mapping from the space of an arbitrary contact fibration to the space of the contact elements of its base (that is, to the space of the projectivized cotangent bundle of the base space).

This mapping transforms fibres into fibres (over the same points). The nondegeneracy of the initial contact structure implies that this mappping is nondegenerate (it is a local diffeomorphism). And it is easy to see that the initial contact structure and the natural contact structure of the contact elements' space agree.

Thus we have obtained a unique local normal form of any Legendre fibration. At the same time we have defined a *natural projective structure* in the fibres of any Legendre fibration.

This projective structure is a contact analogue of the natural affine structure of the fibres of Lagrange fibration in symplectic geometry; this affine structure is the main ingredient of the proof of the Liouville theorem on the invariant tori of integrable Hamiltonian systems.

The projective structure of the Legendre fibres is even better than the affine structure of the Lagrange fibres. Indeed, a diffeomorphism of the base of a Legendre fibration induces a well defined mapping of the fibres (since it acts on the contact elements of the base).

A diffeomorphism of the base of a Lagrangian fibration can be (locally) lifted to a fibred symplectomorphism of the total space, but this lifting is not unique (this ambiguity implies some global annoyances).

According to the above theorem the Legendre singularities (the germs of the triples $L \hookrightarrow E \twoheadrightarrow B$ consisting of a Legendre embedding and of a Legendre fibration) can be modelled by the Legendre submanifolds of a projectivized cotangent bundle of any manifold, say — of the projective space. The Legendre singularity is defined by its front, if it is a hypersurface (and they usually are).

Now it is easy to deduce that all the Legendre singularities are (locally) equivalent to singularities of Legendre transformations of smooth functions, or of the dual hypersurfaces of smooth projective hypersurfaces or the equidistants of smooth hypersurfaces (and so on).

A Legendre singularity is called *simple,* if all the neighbouring Legendre singularities belong to a finite set of Legendre equivalence classes.

A Legendre mapping is Legendre stable if all the neighbouring Legendre mappings are equivalent to the given one. A similar definition for germs allows a small shift of the origin of the germ: any neighbouring Legendre *mapping* has a Legendre equivalent *germ* at a neighbouring point.

THEOREM. *All the simple and stable analytic Legendre singularities are classified by the simple Lie algebras of types* A, D *and* E:

$$\begin{tikzcd}
A_1 & A_2 \arrow[l] & A_3 \arrow[l] & A_4 \arrow[l] & A_5 \arrow[l] & A_6 \arrow[l] & A_7 \arrow[l] & A_8 \arrow[l] & \ldots \arrow[l] \\
 & & & D_4 \arrow[ul] & D_5 \arrow[l] \arrow[ul] \arrow[u] & D_6 \arrow[l] \arrow[ul] \arrow[u] & D_7 \arrow[l] \arrow[ul] \arrow[u] & D_8 \arrow[l] \arrow[ul] \arrow[u] & \ldots \arrow[l] \arrow[ul] \\
 & & & & & E_6 \arrow[ul] \arrow[u] & E_7 \arrow[l] \arrow[ul] \arrow[u] & E_8 \arrow[l] \arrow[ul] \arrow[u] &
\end{tikzcd}$$

Namely, the corresponding fronts are **C**-*diffeomorphic to the corresponding discriminants (the sets of nonregular orbits of the corresponding Weyl groups).*

Example. The Weyl group A_μ is the group generated by the reflections of the space $\mathbf{C}^\mu = \{z \in \mathbf{C}^{\mu+1}: z_0 + \ldots + z_\mu = 0\}$ in the diagonal mirrors $z_i = z_j$. The orbits are the unordered $\mu + 1$-tuples $\{z_0, \ldots, z_\mu\}$, such that $z_0 + \ldots + z_\mu = 0$.

The space of orbits is the space of polynomials

$$z^{\mu+1} + \lambda_1 z^{\mu-1} + \ldots + \lambda_\mu .$$

The irregular orbits correspond to polynomials having multiple roots.

For instance, the set of irregular orbits for A_2 is a semicubical parabola in the plane formed by the polynomials

$$\{z^3 + \lambda_1 z + \lambda_2 = (z+t)^2(z-2t)\} .$$

The discriminant for A_3 is the swallowtail surface

$$\{z^4 + \lambda_1 z^2 + \lambda_2 z + \lambda_3 = (z+t)^2 \ldots\} .$$

THEOREM. *The generic Legendre mappings* $L^n \to E^{2n+1} \to B^{n+1}$ *of Legendre manifolds of dimension* $n < 6$ *are simple and stable at all their points (and hence are described by the preceding theorem).*

The classification of the stable Legendre mappings of any dimension up to Legendre equivalence is equivalent to the classification of families of functions up to "V-equivalence" (= fibred diffeomorphisms of the zero level hypersurfaces).

It is also known that the set of topologically different singularities of generic Legendre mappings remains finite for any finite n (Varchenko, Looijenga). However this topological classification is unknown even for those small values of n between 6 and 11, for which there exists an explicit classification up to smooth Legendre equivalence (this classification is described in the last chapter of volume 1 of the book [AGV]).

In the theory of propagation of waves one encounters, besides the usual wave fronts, a Legendre singularity of higher dimension — its front is the graph of the "multivalued time function", whose level sets are the momentary fronts.

Let us consider the positions of a moving front in different moments of time as a hypersurface in space-time. This hypersurface is called the *big front*. The big front is a front of a Legendre mapping over space-time. The *momentary fronts* are its sections by the isochrones (isochrones are the level sets of the time function in space-time).

To study the perestroikas[1]) of the momentary wave fronts we need to reduce the time function to a normal form in a neighbourhood of a singular point of the big front by a diffeomorphism preserving the big front.

In the case when the big singularity is simple and stable, this can be done very explicitly, using the technique of the invariant theory of Weyl groups (or of Coxeter groups).

The main ingredient is the study of vector fields, tangent to the discriminant. Such vector fields form a module over the algebra of functions. Hence the knowledge of few particular fields, tangent to the discriminants, permits one to construct many diffeomorphisms preserving the discriminant. Using these diffeomorphisms one can reduce the time function in a neighbourhood of the origin of the space of orbits of a Weyl group to a linear normal form. The corresponding linear function on the space of orbits is an invariant of degree 2 (considered as a function on the space of orbits).

For instance, a generic function in a neighbourhood of the most singular point $\lambda = 0$ of the generalized swallowtail $\{\lambda : \exists t : x^{\mu+1} + \lambda_1 x^{\mu-1} + \ldots + \lambda_\mu = (x+t)^2 \ldots \forall x\}$ is reducible to the normal form $\pm\lambda_1$ by a swallowtail preserving diffeomorphism.

[1]) In Russian the word perestroika was always used in this mathematical sense, for instance "Morse surgery" is "Morse perestroika" in Russian. In past translations from the Russian, "perestroika" of wave fronts was called "metamorphosis", but now I may use the international word "perestroika".

One may find in the literature the statement that the local perestroikas of the wavefronts generated by the general Legendre mappings over space-time and of the equidistants of the smooth hypersurfaces are the same. It seems this has never been correctly proved. It is now known (Nay, Tchekanov) that the corresponding statement for the caustic perestroikas is wrong. The fact that the moving Lagrange manifold lies in a (moving) hypersurface of the cotangent bundle space, which is quadratically convex along the fibres, implies some topological restrictions on the local perestroikas of the caustics.

The contact geometry analogues of these results have not yet been formulated (one of the variants deals with the Legendre submanifolds of a hypersurface in the projective cotangent fibration space, which is locally quadratically convex in the sense of the projective structure of the fibres).

The local classification of generic Legendre singularities is the base of a global theory of *Legendre cobordisms* and *characteristic classes.*

Let us consider the projectivized (or the spherized) cotangent bundle $E(M)$ of a manifold M with boundary ∂M. A Legendre submanifold L of M, which is transversal to ∂E, has a "Legendre boundary", which is an immersed Legendre submanifold of $E(\partial M)$. It is defined by "section and projection": first we intersect L with the hypersurface ∂E, and then we project the intersection along the characteristics of ∂E to the space of characteristics, which is $E(\partial M)$. The projection has dimension $\dim L - 1$ and is a Legendre submanifold of $E(\partial M)$.

This Legendre boundary construction gives birth to many cobordism theories since we can consider oriented or non-oriented bases and immersed Legendre manifolds, formed by cooriented or noncooriented contact elements or by jets of functions on manifolds with boundary.

Example 1. The group of cobordism classes of (a) oriented, (b) nonoriented Legendre submanifolds in the space of cooriented contact elements of the plane is isomorphic to (a) the group of integers, $\mathbf{Z}$, the generator being an eight-shaped curve with 2 cusps at the top and at the bottom; (b) to the trivial group.

Example 2. The cobordism of the oriented generator to zero shows that this generator is the Legendre boundary of a Legendre Möbius band over a halfplane M. This construction defines a Legendre embedding of a Klein bottle in $\mathbf{R}^5$. (See [A2].)

For more details on the Legendre cobordisms and characteristic classes consult the book [Va] by V.A. Vassilyev.

Example 3 (M. Audin). The classes of nonoriented Legendre cobordisms in the spaces of 1-jets of functions in the spaces $\mathbf{R}^n$ form a skew-commutative graded ring, which is isomorphic to the graded ring $\mathbf{Z}_2[x_5, x_9, x_{11}, \ldots]$ of polynomials with coefficients in $\mathbf{Z}_2$ and with arguments x_k of odd degrees $k \neq 2^r - 1$.

In the oriented case the ring is isomorphic to the exterior algebra over $\mathbf{Z}$ with generators of degrees $1, 5, 9, \ldots, 4n + 1, \ldots$ mod torsion.

The proofs are based on the Eliashberg reduction of the problem to the calculation of the homotogy groups of the Thom spectra of the tautological bundles over the Lagrangian Grassmannians (the details are in the Eliashberg paper [El]).

On the other side the classification of Legendre singularities generates a complex, whose cells are singularity types and whose boundaries are defined by the adjacency of the singularities. The initial parts of these complexes were calculated by V.A. Vassilyev (see his book [Va]). The cohomology of these complexes defines Legendre characteristic classes (the simplest of them is the Maslov class). These classes can be generated also by the corresponding universal spaces (the Lagrangian Grassmannians $U(n)/O(n)$).

But the information on the singularities' coexistence, compressed in the Vassilyev complexes of singularities and of multisingularities is not reduced to the calculation of the characteristic numbers in terms of the singularities.

Example. The number of A_3 points on a generic closed Legendre surface immersed in $J^1(M^2, \mathbf{R})$ is always even. The number of intersections of the strata (A_1A_2), (A_1A_4), (A_2A_4), (A_1A_6), $(A_1A_2A_4)$ of the front are (mod 2) characteristic numbers for the Legendre mappings.

The number of singularities of any given type on the Legendre boundary is even. For the Legendre boundary of an oriented manifold the numbers of singularities E_6, (or E_7 or E_8) counted with some sign convention, are equal to zero.

For Legendre immersions in the space of 1-jets of functions, Vassilyev has defined orientation rules, for which $\# A_5 = 0$ (the number of A_5 singularities on a closed oriented Legendre 4-manifold is equal to zero), $\# A_6 = \# E_6$, $\# E_7 + 3\# E_8 = 0$. The class A_5 defines a cohomology class in the Vassilyev complex, but it is not realizable by a Legendre immersion.

The topology of Legendre immersions and embeddings is far from being settled.

§5. LEGENDRE VARIETIES AND THE OBSTACLE PROBLEM

According to a well known principle of A. Weinstein, everything in symplectic geometry "is" a Lagrangian manifold.

In contact geometry "everything" is a Legendre manifold. But the important examples suggest that in many applications smooth Lagrangian (Legendre) manifolds should be substituted by singular Lagrangian (Legendre) varieties.

In principle the translation of the general theory to the case of singular varieties (and even of the "schemes" of algebraic geometry) is routine work. But I believe that a more natural and useful notion of the Lagrangian (Legendre) singular varieties would be a generalization of the classification of the Givental "triads", presented below, rather than a theory of Legendre ideals of the hierarchy of degeneracies in the generating families.

Let us consider a medium, containing an obstacle (i.e. a manifold with a boundary). The fronts are the hypersurfaces, equidistant from a given one. For instance, if the obstacle is bounded by a plane curve, the fronts are its evolvents (involutes). Hence the following is a higher-dimensional generalization of the Huygens' involute theory.

A smooth front may acquire singularities while travelling through a smooth medium, but its Legendre manifold remains non-singular. At an obstacle, however, even the Legendre manifold may become singular. These singular *Legendre varieties* are singularly related to the irreducible finite-dimensional sl_2-modules. Namely, the singularities of the Legendre varieties at generic obstacles are diffeomorphic to those of the varieties of binary forms (or of polynomials in one variable) admitting roots of high multiplicity.

The simplicity of the final result is rather misleading: the polynomials and even their degrees are hidden. Even though they are known to exist it is still difficult to find them from geometrical considerations.

The relation of the obstacle problem to $sl(2)$ modules was discovered in 1981-82 as a result of a series of works featuring geometrical observations based on the resemblance of bifurcation diagrams, occurring in different theories, strange cancellations of many terms in long calculations, due to some properties of the varieties of polynomials with multiple roots, which seems to be new for the algebraists, and new concepts in symplectic and contact geometry, namely the "triads" of Givental, describing families of rays and of fronts at obstacle points.

Let us consider a smooth hypersurface ∂M in Euclidean space $M = \mathbf{R}^n$. Let us consider the length S of the shortest path from a fixed set to a variable end point, avoiding the obstacle, bounded by ∂M. The study of singularities of S as a function of the end point leads to the following problem.

Let us consider a family of geodesics on ∂M, orthogonal to some hypersurface of ∂M. The straight lines, tangent to these geodesics, define an $(n-1)$-parameter family of rays in $\mathbf{R}^n$, namely the family of all normals to some front hypersurface in $\mathbf{R}^n$. The problem is to study the singularities of these fronts.

Example 1. Let the obstacle be bounded by a generic plane curve ($n=2$). The fronts are the *involutes* of the curve. They have singularities of order 3/2 at generic points of the curve (Huygens). A generic curve may have some inflection points. A calculation shows that the fronts have singularities of order 5/2 at the points of the inflectional tangent. These singularities are related to the rather mysterious appearance of H_2 — of the symmetry group of a pentagon (at the point of the inflection it is replaced by the symmetry group H_3 of an icosahedron, but it is rather difficult to see this icosahedron in the neighbourhood of the inflection point with the naked eye).

Example 2. Let the obstacle be bounded by a generic surface in 3-space. The fronts are surfaces with cuspidal edges. These edges are of order 3/2 at generic points of the boundary surface. Our one-parameter family of geodesics covers a domain in this surface. The geodesic direction may become asymptotic along some curve in this domain. The rays tangent to the geodesics at the points of this curve have asymptotic directions. The fronts' singularities at the points of an asymptotic ray are edges of order 5/2, unless the ray is "bi-aymptotic" (this may happen at some points of our curve).

The theory described below explains the contact geometry of these complicated singularities and of their higher dimensional counterpart in terms of the theory of invariants of sl_2.

Definition. A *contact triad* (H, L, l) consists of

(i) a noncharacteristic hypersurface H in a contact manifold,

(ii) a Legendre submanifold L in the same contact manifold,

(iii) a smooth hypersurface l in L

such that the hypersurface H is tangent to the Legendre manifold L with first order of tangency at every point of l.

We shall study the germ of a triad at a point 0 of l.

Definition. The *Legendre variety, generated by the triad* at 0 is the image of the germ of l at 0 by the projection of H onto its space of characteristics.

Example. Let us consider the family of geodesics on a hypersurface $\partial M \subset M = \mathbf{R}^n$, consisting of all geodesics, normal to a surface of codimension 1 in ∂M. We shall associate to this family a contact triad.

Let H be the hypersurface of light contact elements of the space-time $\mathbf{R}^n \times \mathbf{R} = \{q, t\}$, defined by $dt = pdq$, $p^2 = 1$. This hypersurface is the first element of the triad.

Let $s \colon \partial M \to \mathbf{R}$ be the time function, defining the family of geodesics on ∂M (the geodesics are orthogonal to the surfaces $s = \text{const}$, and $(\nabla s)^2 = 1$ on ∂M). The graph of s is a codimension 2 submanifold of space-time. Let us consider the set of all space-time contact elements, tangent to this graph. This set is a Legendre manifold and it will serve as the L of the triad.

THEOREM. *The hypersurface H is (first order) tangent to the Legendre manifold L along a submanifold l of codimension 1 in L, consisting of all contact elements, belonging to L, which contain the normal to $\partial S \times t$ in $\mathbf{R}^n \times t$.*

The Legendre variety, generated by this triad, is formed by those contact elements of $\mathbf{R}^n$, which are tangent to the same front for the obstacle problem with boundary ∂M (and initial condition s).

The theorem follows almost immediately from the analysis of the hexagonal diagram in § 3. For more details see [A5] and [A6].

We will now construct a series of examples of triads, providing normal forms for the germs of generic triads at all their points. This implies, for instance, normal forms of singularities of the Legendre varieties, consisting of all contact elements tangent to a front for a generic obstacle in a Euclidean or Riemannian space.

We start with the natural sl_2-invariant contact structure of the projective space of the 0-dimensional hypersurfaces of degree $d = 2n - 1$ on the projective line (§ 1, example 7).

Let us consider maps of the space of polynomials of the form

$$X_{2n-1} + q_2 X_{2n-2} + \dots + q_n X_n - p_n X_{n-1} + \dots \pm p_1 ,$$

where $X_j = x^j/j!$, with the contact structure $\alpha = 0$, where

$$\alpha = p'dq' - q'dp' - dp_1 , \quad p' = (p_2, \dots, p_n) , \quad q' = (q_2, \dots, q_n) .$$

The group of translations of polynomials along the x-axis acts on the space of these polynomials and preserves its contact structure. Let v be the corresponding vector field. We define the *neutral surface* H by the equation $\alpha \mid v = 0$. An explicit calculation gives the equation of the neutral hypersurface

$$H: K \equiv p_2 + p_3q_2 + \dots + p_nq_{n-1} + q_n^2/2 = 0$$

(known essentially to Hilbert).

This formula implies the obvious

LEMMA. *The triple* $H(K=0)$, $L(p=0)$, $l(p=q_n=0)$ *is a contact triad.*

The Legendre variety Σ, generated by this triad, consists of those polynomials

$$X_{2n-1} + q_3X_{2n-3} + \dots + q_nX_n - p_nX_{n-1} + \dots + (-1)^n p_1 ,$$

where $X_j = x^j/j!$ and $p_2 = -(p_4q_3 + \dots + p_nq_{n-1} + q_n^2/2)$, which have a root of multiplicity greater than n; the contact structure is defined by the 1-form

$$\alpha = p''dq'' - q''dp'' - dp_1 , \quad p'' = (p_3, \dots, p_n) \quad q'' = (q_3, \dots, q_n) .$$

The Legendre variety Σ^m of dimension $m = n - 2$ thus defined will be called the *Givental Legendre variety*. It lives in a contact space of dimension $2m + 1$.

These varieties (and their Lagrangian projections to symplectic spaces, also studied by Givental) have remarkable properties, both as algebraic varieties and as contact (or symplectic) space subvarieties.

Let us first describe them as algebraic varieties. We start with the tower of spaces of polynomials in one variable x, equipped with the projection given by the derivative $D = (n+1)^{-1}(d/dx): \mathbf{C}^n \to \mathbf{C}^{n-1}$,

$$\mathbf{C}^n = \{x^{n+1} + A_1x^{n-1} + \dots A_n\} ,$$

(one may consider as well the tower of the spaces of polynomials $x^{n+1} + A_0x^n + \dots$ or even $Ax^{n+1} + A_0x^n + \dots$).

Let us consider a root of multiplicity m of a polynomial of degree d.

The number $d - m$ is called the *comultiplicity* of a root. The spaces of polynomials are "stratified" according to the comultiplicities. We denote the set of polynomials $x^{n+1} + A_1x^{n-1} + \dots + A_n$ having a root of comultiplicity at most m by $\Sigma_m(n) \subset \mathbf{C}^n$. $\Sigma_m(n)$ is an algebraic variety of dimension m.

Example. $\Sigma_1(2)$ is the discriminant curve in the plane of cubical polynomials $x^3 + A_1x + A_2$, $\Sigma_1(3)$ is the cusped edge of the swallowtail, $\{x^4 + B_1x^2 + B_2x + B_3 = (x+t)^3 \dots\}$.

THEOREM. *The derivation mapping* $D\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ *preserves the comultiplicities, provided that the multiplicity is greater than one:* $D\Sigma_m(n) = \Sigma_m(n-1)$ *for* $m < n$.

Moreover, this mapping is a diffeomorphism, provided that Σ_m *has no selfintersections, namely for* $n > 2m$.

Example 1. The cusped edge of the swallowtail, $\Sigma_1(3)$, projects diffeomorphically on the plane semicubical parabola $\Sigma_1(2)$.

Example 2. The surface $\Sigma_2(4) \subset \mathbf{C}^4$ projects onto the usual swallowtail surface $\Sigma_2(3)$, but this mapping is not one-to-one, since generic points of the selfintersection line of the swallowtail surface have two counterimages.

If we start with the generalized swallowtail $\Sigma_m(m+1)$ of dimension m in $\mathbf{C}^{m+1}$ and go up the storeys of the tower, we obtain a sequence of m-dimensional varieties and of projections

$$\dots\ \Sigma_m(2m+1) \to \Sigma_m(2m) \to \dots \to \Sigma_m(m+1)\,.$$

The sequence stabilizes at the floor of $\Sigma_m(2m) \subset \mathbf{C}^{2m}$, where the last selfintersection disappears and the variety becomes homeomorphic to $\mathbf{C}^m$ (A.B. Givental).

Example. The ordinary swallowtail $\Sigma_2(3)$ is stabilized at the next floor, where lives the *open* (or "*unfurled*") *swallowtail* $\Sigma_2(4)$. This surface is stable ($\Sigma_2(4) \approx \Sigma_2(5) \approx \dots \approx \Sigma_2(\infty)$).

The importance of the open swallowtail for variational problems was discovered in 1981 (see [A3] and [A4]).

Remark. The diffeomorphism $\Sigma_m(2m+1) \to \Sigma_m(2m)$ is induced by a section of the mapping $D\colon \mathbf{C}^{2m+1} \to \mathbf{C}^{2m}$, which is a paraboloid. The equation of this paraboloid was found by Hilbert (1893). It follows from his

LEMMA. *Let*

$$x^n + \lambda_1 x^{n-1} + \dots + \lambda_n = (x-a)^{k+2}(x^k + a_1 x^{k-1} + \dots + a_k)\,.$$

Then

$$2n!\lambda_n = \Sigma(-1)^{i+1} i!(n-i)!\lambda_i\lambda_{n-i} \qquad 1 \leqslant i < n\,.$$

The next floors also admit parabolic sections. Let us define the operators $G^{(s)}$ by the formula

$$G^{(s)}[F] = (d^sF/dx^s)/(s-r)!$$

where r is fixed and where the factorials of negative numbers are defined by a truncation at some sufficiently remote place, say

$$(s-r)! = (s-r) \dots (1-r) .$$

THEOREM. *If* $F=(x-a)^{r+1}(x^{m-1}+\dots)$, *where* $r>m$, *then*

$$\Sigma(j-i)G^{(i)}[F]G^{(j)}[F] = 0$$

(summation over $i+j=r+m, 0\leqslant i\leqslant m$*).*

This theorem implies the stabilization property.

For further details on the stabilization and on the general theory of unfurled swallowtails see [G1] and [A6].

For instance, the modules of vector fields, tangent to the unfurled swallowtails have been studied by Givental, who has proved the

THEOREM. *Any germ of a holomorphic vector field, tangent to the generalized swallowtail* $\Sigma_m(m+1)$ *at the origin, is the projection of a germ of a vector field in* $\mathbf{C}^{2m}$ *tangent to its stabilization* $\Sigma_m(2m)$.

Any polynomial vector field, tangent to $\Sigma_k(n)$, may be represented as a sum of a vector field, tangent to the swallowtail $\Sigma_{n-1}(n)$ and of vector fields whose projections to $\mathbf{C}^{n-1}, \dots, \mathbf{C}^{k+1}$ are vector fields tangent to the projections $\Sigma_k(n-1), \dots, \Sigma_k(k+1)$ of the original variety.

The Givental theory implies that the unfurled swallowtail $\Sigma_m(2m)$ is a Lagrangian subvariety of the space of polynomials $x^{2m+1}+a_1x^{2m-1}+\dots+a_{2m}$ equipped with its natural symplectic structure

$$\Sigma(-1)^i i!j!\,da_i \wedge da_j , \quad i+j=2m-1 .$$

(This structure is natural in the sense that it is naturally derived from the sl_2-invariant symplectic form on the space of binary forms of odd degree.)

Example. The ordinary (two-dimensional) unfurled swallowtail $\Sigma_2(4) = \{x^5+Ax^3+Bx^2+Cx+D=(x+t)^3\dots\}$ is a Lagrangian subvariety for the natural symplectic structure

$$3dA \wedge dD - dB \wedge dC .$$

The unfurled swallowtails describe the singularities and the perestroikas of the duals of the projective space curves. Let $\phi: \mathbf{R} \to \mathbf{R}^3$ be a smooth map. We call (a, b, c) its *type*, if the first of its derivatives which is nonzero at the origin, is the a-th one, the first noncolinear with it — the b-th, the first noncoplanar with them — the c-th.

The curves of types $(a_1, \dots, a_n)$ in $\mathbf{R}^n$, $\mathbf{RP}^n$, $\mathbf{C}^n$ or $\mathbf{CP}^n$ are defined by a similar construction of an osculating flag. Any curve of a finite type has an osculating hyperplane at every point.

Definition. The *dual curve* $\phi^\vee : \mathbf{R} \to \mathbf{RP}^{n\vee}$ of a curve $\phi : \mathbf{R} \to \mathbf{RP}^n$ is the set of osculating hyperplanes of ϕ. Of course, $\phi^{\vee\vee} = \phi$ and the dual type of $(a_1, \dots, a_n)$ is $(a_n - a_{n-1}, \dots, a_n - a_1, a_n)$.

The hierarchy of the smooth $(a_1 = 1)$ curves in $\mathbf{R}^3$ starts with

$$\begin{array}{ccccccc} (1,2,3) & \leftarrow & (1,2,4) & \leftarrow & (1,2,5)\ \text{(biflattening)} & \leftarrow & \dots \\ \text{(ordinary point)} & & \text{(flattening)} & & & & \\ & & & \nwarrow & & & \\ & & & & (1,3,4)\ \text{(inflection)} & \leftarrow & \dots \end{array}$$

The codimension of the type is $c = \Sigma(a_i - i)$.

The set of all the hyperplanes, containing the tangent line of a projective curve ϕ, forms a developing hypersurface in the dual space, having a cusped edge $\phi^\vee$. We call this developing hypersurface *the front of* ϕ. (This is a particular case of the general definition of the front of a submanifold of projective space as of fronts of the corresponding Legendre mapping. We start with any submanifold M in $\mathbf{RP}^n$, and construct the Legendre submanifold L formed by the contact elements of $\mathbf{RP}^n$, tangent to M, and we project L to the dual space $\mathbf{RP}^{n\vee}$ along the fibres of the Legendre fibration:

$$M^k \qquad L^{n-1} \to PT^*\mathbf{RP}^n = PT\mathbf{RP}^{n\vee} \twoheadrightarrow \mathbf{RP}^{n\vee}.$$

The resulting Legendre mapping $L^{n-1} \to \mathbf{RP}^{n\vee}$ is called the *frontal mapping of M*).

Example. Let the curve ϕ have a simplest flattening $(1, 2, 4)$. Then the curve $\phi^\vee$ has a singularity of type $(2, 3, 4)$ like (t^2, t^3, t^4). The tangent lines sweep out the usual swallowtail which is the front of ϕ.

A generic curve in $\mathbf{R}^3$ has isolated flattening points but has no more complicated degeneracies. Hence a front of a generic space curve is a surface with a cusped edge and with isolated swallowtails.

In a 1-parameter family of curves in $\mathbf{R}^3$ the bi-flattening points become unavoidable (for some exceptional values of the parameter). The family of dual curves in a family of dual spaces forms a surface in a 4-space.

THEOREM (0. Shcherback). *This surface is locally diffeomorphic to the ordinary unfurled swallowtail and its decomposition into dual curves is dif-*

feomorphic to the decomposition of the space of polynomials $x^5 + Ax^3 + Bx^2 + Cx + D$ *having a triple root into the "isochrones", defined by the equation* $A = const.$

The inflection points give rise to the surface $\{x^4 + Ax^3 + Bx^2 + Cx + D$ has a multiple root$\}$ and to the "isochrones" $B =$ const.

The general theory of Shcherbak is described in his paper [Sh1].

For instance, he has proved the following theorems.

1. A front of a curve, dual to a generic smooth space curve (that is, the union of the tangents of the smooth curve) has at the (1, 2, 4) inflection point of the original smooth curve a "folded Whitney umbrella", locally diffeomorphic to the germ of the surface $x^2y^3 = z^2$ at the origin.

2. The singularities of the front of a generic smooth curve in $\mathbf{RP}^n$ are locally diffeomorphic to the discriminants A_k, i.e. to the products of generalized swallowtails and of smooth manifolds. The union of the front of a curve with the hyperplane, dual to a point of the initial curve, is locally diffeomorphic to a discriminant of Lie algebra B_k.

3. In typical one-parameter families of curves in $\mathbf{RP}^n$ there exist unavoidably isolated points of types $(1, \ldots, n-1, n+2)$ and $(1, \ldots, n-2, n, n+1)$. The corresponding fronts' perestroika patterns are

$$\{x^{n+2} + \lambda_1 x^n + \ldots + \lambda_{n+1} = (x+t)^2 \ldots\}, \quad \lambda_1 = \text{const.};$$
$$\{x^{n+1} + \lambda_1 x^n + \ldots + \lambda_{n+1} = (x+t)^2 \ldots\}, \quad \lambda_2 = \text{const.}$$

4. Any stable Legendre singularity of corank m is Legendre equivalent to a germ of a frontal mapping of an m-dimensional submanifold of a projective space. The frontal mappings of the generic space curves have only A_k singularities. The perestroikas of the Legendre mappings of corank m in the generic families with finite dimensional parameter spaces are Legendre equivalent to the perestroikas of the frontal mappings of the m-dimensional submanifolds of projective spaces.

The latest development of the contact geometry of projective space curves is due to M. Kazarjan (1985; see his paper [Ka]). In his theory the higher codimension degenerations are classified by their Young diagrams, and related to the singularities of the Schubert cell decomposition of a Grassmannian manifold.[1])

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[1]) The Schubert cells of flag manifolds are related to Tchebychev systems and to nonoscillating linear ODE, as was discovered by B.Z. Shapiro (1985).

The Kazarjan theory introduces some new remarkable bifurcation diagrams. For instance, the Young diagram (2, 2) corresponds to a union of two Whitney umbrellas, tangent cubically along two lines. One of the ways of visualizing this bifurcation diagram is due to Shcherbak. Let us consider a generic 1-parameter family of projections of a smooth curve to a plane. For some isolated values of the parameter the projection has a cusp. The corresponding perestroika is of type "from γ to u". The projections may also be described as the generic sections of a Whitney umbrella by " isochrones" (parallel planes).

Now let us add to each of these plane curves their inflectional tangents. These tangents sweep out a surface in 3-space. This surface is the second Whitney umbrella. The two umbrellas together from the Kazarjan bifurcation diagram. The general theory of such diagrams and of their applications to different problems of calculus, optics and so on may be found in Kazarjan's paper quoted above and in the 2 volumes of the Springer Mathematical Encyclopaedia, devoted to singularities (the first of these is the volume "Dynamical systems — 6", Moscow VINITI 1988, but we need the second, VINITI 1989, "Dynamical systems – 8", Encyclopaedia vol. 39).

Let us now return to the obstacle problem. Comparing the explicit formula defining the Givental Legendre variety and the definition of the unfurled swallowtails, we obtain

THEOREM. *The Givental Legendre variety of dimension* m *is diffeomorphic to the unfurled swallowtail* $\Sigma_m(2m) = \Sigma_m(\infty)$.

Example. The 2-dimensional Givental Legendre variety lives in the contact 4-space of polynomials

$$x^7/7! + q_3x^5/5! + q_4x^4/4! + p_4x^3/3! - p_3x^2/2! + ux - p_1,$$

where

$$u = -(q_3p_4 + q_4^2/2), \qquad \alpha = p_3dq_3 + p_4dq_4 - q_3dp_3 - q_4dp_4 - dp_1,$$

and consists of polynomials, which have a root of multiplicity 5 or greater. It is diffeomorphic to $\Sigma_2(4)$.

THEOREM. *The standard contact triads defining the Givental Legendre varieties, are stable (as germs of contact triads considered up to contact equivalence). The germs of generic contact triads are contactoequivalent to the germs of the standard ones.*

COROLLARY. *The Legendre varieties defined by generic contact triads are locally contactomorphic to the Givental Lagrange varieties.*

COROLLARY. *The germs of Legendre varieties, formed by contact elements, tangent to a front in the generic obstacle problem are contact stable and contact equivalent to the germs of the Givental Legendre varieties (and hence are diffeomorphic to the germs of the unfurled swallowtails).*

Returning to the hexagonal diagrams of § 3, we can now find the normal forms of the Legendre varieties describing the "multivalued time functions".

THEOREM. *The 1-graph (the set of 1-jets) of the time function in a generic obstacle problem in* $\mathbf{R}^m$ *is locally contactomorphic to the Legendre variety formed by the polynomials* $(x-\xi)^{m+2}(x^{m-1}+\ldots)$ *in the space of polynomials of degree* $n = 2m-1$

$$x^n + a_1 x^{n-1} + \ldots + a_n ,$$

equipped with its natural (sl_2*-invariant) contact structure* $\Sigma i!j!(-1)^i a_i\, da_j = 0$ *(where* $i+j=n$, $a_0=1$*).*

The time function restriction to this variety is $\pm a_1 + \text{const}$*; the points of the same ray correspond to the translations of a polynomial along the* x *axes.*

COROLLARY. *The variety of the contact elements of a generic front in the obstacle problem in* $\mathbf{R}^m$ *is locally contactomorphic to the Legendre variety.*

$$\{\lambda : \exists \xi : x^n + \lambda_{n-1}x^{n-1} + \ldots + \lambda_0 = (x-\xi)^m (x^{m-1} + m\xi x^{m-2} + \ldots)\ \forall x\}$$

in the space of polynomials of degree $n = 2m-1$ *equipped with the perverse contact structure*

$$d\lambda_{n-1} = \Sigma i!j!(-1)^i (i+2)\lambda_i d\lambda_j, \quad 0 \leqslant i \leqslant n-2, \quad i+j = n-2.$$

Now to derive the singularities of the fronts and of their perestroikas in the obstacle problem we have to project the above normal forms to the base of a Legendre fibration (for instance, from the space J^1 of 1-jets of functions to the space J^0 of 0-jets, where the ordinary graph of the time function lives). This is unfortunately a difficult problem (it is discussed in Givental's thesis, and in his paper [G2]).

The classification of the generic front singularities in the obstacle problem was obtained by O.P. Shcherbak in another way in 1984 (see, for instance, Uspekhi Math. Nauk 1984, vol. *39,* N 5, p. 256[1])). Unfortunately, the final text of his proofs appeared only after he died in 1985.

I am very grateful to I.G. Shcherbak and to A.B. Givental who have prepared O.P. Shcherbak's manuscript "Wave fronts and reflection groups" for publication. It has finally appeared in [Sh3].

The main discoveries of Shcherbak in this paper are the local diffeomorphisms of the fronts and the graphs of the multivalued time function in the generic obstacle problem to the discriminants of the noncrystallographic Coxeter groups H_2, H_3, H_4.

H_2 is the symmetry group of a pentagon. Its orbit space is $\mathbf{C}^2$, and the irregular orbits form a discriminant curve with a singularity $x^2 = y^5$. It was perhaps known to Huygens and it is written explicitly in the book of L'Hospital (1696), that this singularity appears at the inflection tangent to a generic plane curve as a singularity of the involute (that is, of the generic front in a two-dimensional obstacle problem).

H_3 is the symmetry group of an icosahedron. The orbit space is $\mathbf{C}^3$ and the discriminant surface has been studied by O.V. Lyashko with the help of a computer. A.B. Givental (1982) recognized in this picture the graph of the multivalued time function of the plane obstacle problem, which I had shown him a year before. Then O.P. Shcherbak proved the Givental conjecture: the germ of the multivalued time function at the generic inflection point of the obstacle is locally diffeomorphic to the surface of irregular orbits of H_3. One may find the proofs in the papers [Ly] and [Sh2].

H_4 is the symmetry group of a convex polyhedron with 120 vertices in $\mathbf{R}^4$. To describe this polyhedron we start from the rotation group of an icosahedron which contains 60 elements. The double covering $S^3 \to SO(3)$ lifts this group to a subgroup of 120 elements in S^3. Those 120 points of S^3 form the vertices of our polyhedron.

In his study of the singularities of the fronts and time function graphs in the obstacle problem, O.P. Shcherbak has found among other things the discriminant of H_4. Namely it is the singularity of the graph of the multivalued time function at some "focal" point of a tangent line to a geodesic of the family of geodesics on the surface of the obstacle (defined by the initial condition). The tangent line itself is very special: it is an asymptotic line of the surface in one of its parabolic points (for a generic family of geodesics the direction of the geodesics changes along the parabolic line of the surface and

[1]) "Uspekhi" are translated by the London Mathematical Society as "Russian Math. Surveys". But some pages of the Uspekhi contain news and announcements and hence are not translated.

consequently becomes asymptotic at some isolated points of the parabolic line; these points depend on the family).

The proof depends on a classification of the families of functions with critical points of only even multiplicity. The simple germs of this type are

$$A'_{2k}: y^2 + \int_0^x (u^k + q_1 u^{k-2} + \ldots + q_{k-1})^2 du + q_k ;$$

$$D'_{2k}: \int_0^y (u^{k-1} + q_1 u^{k-3} + \ldots + q_{k-3} u + x)^2 du + q_{k-2} x^2 + q_{k-1} x + q_k ;$$

$$E'_6: x^3 + y^4 + q_1 y + q_2 y + q_3 ;$$

$$E'_8: x^3 + y^5 + q_1 y^3 + q_2 y^2 + q_3 y + q_4 ;$$

$$E''_8: x^3 + \int_0^y (u^2 + q_1 x + q_2)^2 du + q_3 x + q_4 .$$

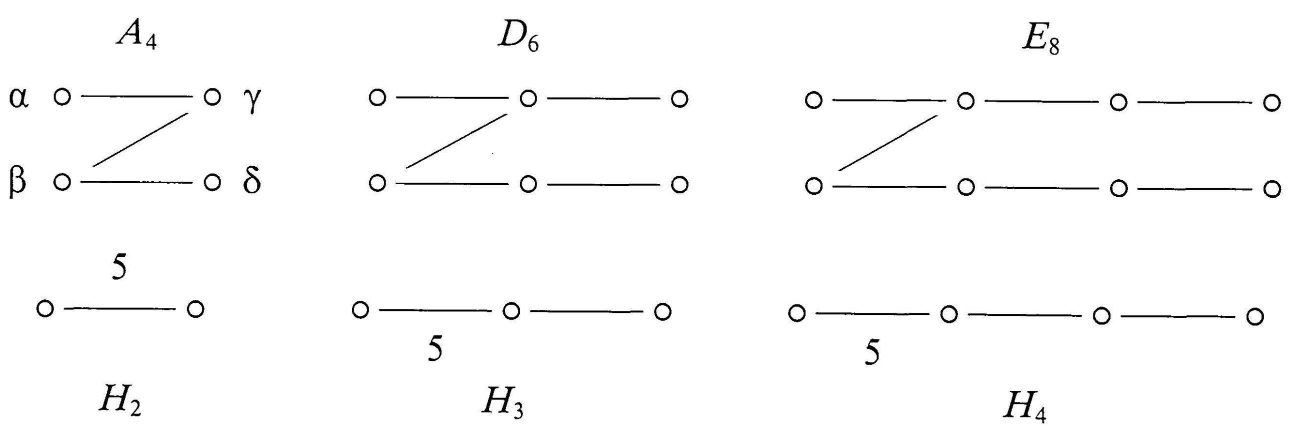
The front of the family is the set of parameters q, such that 0 is a critical value.

THEOREM. *The optical length in the generic obstacle problem in* $\mathbf{R}^3$ *(considered as a family of functions of the initial point depending on the final point as on a parameter) has only the simple critical points of the preceding list. Hence the graph of the multivalued time function ("the big front") is locally diffeomorphic to the Cartesian product of the front of one of the families* $A'_2, A'_4, A'_6, D'_6, E'_6, D'_8, E'_8, E''_8$ *with a non-singular manifold.*

From A'_4 one obtains the 5/2 singularity H_2, from D'_6 the icosahedral discriminant H_3. This is the singularity of the front at the generic points of the surface, where the geodesic has an asymptotic direction.

The discriminant of the group H_4 is diffeomorphic the front of the family E''_8

The paper of Shcherbak contains a lot of information on these and other Coxeter groups. It is interesting to note that the "foldings" $A_4 \to H_2$, $D_6 \to H_3$, $E_8 \to H_4$ may be defined by the unusual forms of the Dynkin diagrams



Let us first describe the folding $A_4 \to H_2$. The reflections, corresponding to α and to β commute and their product is an element of the A_4 reflection group. In the same way $\gamma\delta$ defines another element, and these two elements generate a subgroup in A_4. This subgroup is a representation of the pentagon symmetry group H_2 in A_4. It is reducible and $\mathbf{R}^4$ is decomposed into a direct sum of two 2-planes invariant under H_2. This construction of an irrational subspace in the space $\mathbf{R}^4$ with a lattice A_4 everything being invariant under the 5-fold symmetry of H_2, allows us to define in $\mathbf{R}^2$ the quasiperiodic Penrose tilings having H_2 symmetry (for the details see [A7]).

The same way the folding $D_6 \to H_3$ generates a subspace $\mathbf{R}^3 \subset \mathbf{R}^6$, invariant under the action preserving the D_6-lattice of the icosahedral symmetry group H_3. This way we construct quasicrystals in $\mathbf{R}^3$ with the icosahedral symmetries.

Finally, the construction of H_4 from E_8 defines in $\mathbf{R}^4$ quasicrystals with the 120×120 symmetries of H_4.

Since the spaces $\mathbf{R}^4$, $\mathbf{R}^6$, $\mathbf{R}^8$ and their lattices A_4, D_6, E_8 may be interpreted as the homology of the corresponding Milnor fibres with $\mathbf{R}$ or $\mathbf{Z}$ coefficients, we obtain some special functions, associated to H_2, H_3, H_4 (generalizing the Airy function associated to A_2, the Piercy function, associated to A_3 and so on, see [VC]).

There exists one more series of noncrystallographical Coxeter groups, $I_2(p)$.

A.B. Givental has discovered a problem in contact geometry, whose solutions are in a one-to-one correspondence with the Coxeter Euclidean reflection groups. This is the problem of the Legendre classification of the simple stable Legendre singularities, whose Legendre varieties are diffeomorphic to the products of curves (or of at most one singular curve with a smooth manifold). His treatment of the series $I_2(p)$ is based on the multiple folding of an A-diagram (see [G2]).

REFERENCES

[A1] ARNOLD, V.I. Small denominators and problems of stability of motion in classical and celestical mechanics. *Russian Math. Surveys 18, 6* (1963), 85-191.

[A2] —— Lagrange and Legendre cobordisms I and II. *Functional Anal. Appl. 14* (1980), 167-177 and 252-260.

[A3] —— Lagrangian manifolds with singularities, asymptotic rays and the open swallowtail. *Functional Anal. Appl. 15* (1981), 235-246.

[A4] —— Evolution of singularities of potential flows in collision-free media and caustics metamorphosae in 3-space. *Trudy Semin. im. I.G. Petrovskogo 8* (1982), 21-57. (English translation: *J. Sov. Math. 32* (1986), 229-258).

[A5] —— Singularities of Legendre varieties, of evolvents and of fronts at an obstacle. *Ergodic Theory and Dynam. Syst. 2* (1982), 301-309.

[A6] —— Singularities in the calculus of variations. *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., VINITI, Moscow, 22* (1983), 3-55 (English translation: *J. Sov. Math. 27* (1984), 2679-2713).

[A7] —— Remarks on the quasicrystallic symmetries. *Physica D, 33* (1988), 21-25.

[A8] —— On surfaces defined by hyperbolical equations. *Matem. Zametki 44, 1* (1988), 3-18 (English translation: *Math. Notes 44* (1988)).

[A9] —— On the interior scattering of wawes defined by hyperbolic variational principles. *J. of Geometry and Physics,* 1989.

[AGV] ARNOLD, V.I., S.M. GUSEIN-ZADE and A.N. VARCHENKO. *Singularities of differentiable maps,* I and II (in Russian). Nauka, Moscow, 1982 and 1984. English translation: Birkhäuser, Boston, 1985 and 1988.

[Co] CONN, Jack F. Normal forms for analytic Poisson structures. *Annals of Math. 119* (1984), 577-601.

[El] ELIASHBERG, J. Cobordisme des solutions de relations différentielles. *Séminaire Sud-Rhodanien de Géométrie I,* Travaux en cours, Paris, Hermann, 1984, 17-31.

[G1] GIVENTAL, A. B. Manifolds of polynomials having a root of fixed multiplicity, and the generalized Newton equation. *Functional Anal. Appl. 16* (1982), 10-14.

[G2] —— *Itogi Nauki Tekh., Ser. Sovrem, Probl. Mat., VINITI, Moscow, 33* (1988). (English translation: *J. Sov. Mat.*)

[GV] GIVENTAL, A.B. and A.N. VARCHENKO. Mapping of periods and intersection form. *Functional Anal. Appl. 16* (1982), 83-93.

[Ka] KAZARJAN, M. *Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., VINITI, Moscow, 33* (1988). (English translation: *J. Sov. Math.*).

[Ki] KIRILLOV, A. A Local Lie algebras. *Russian Math. Surveys 31, 4* (1976), 55-75.

[La] LANDIS, E.E. Tangential singularities. *Functional Anal. Appl. 15* (1981), 103-114.

[Ly] LYASHKO, O.V. Classification of critical points of functions on a manifold with singular boundary. *Functional Anal. Appl. 17* (1983), 187-193.

[Ma] MARTINET, Jean. Sur les singularités des formes différentielles. *Ann. Inst. Fourier 20, 1* (1970), 95-178.

[Me] Melrose, R.B. Equivalence of glancing hypersurfaces, II. *Math. Ann. 255* (1981), 159-198.

[Sh1] Shcherbak, O.P. Projectively dual space curves and Legendre singularities. *Trudy Tbiliss. Univ. 232-233* (1982), 280-336. (English translation: *Sel. Math. Sov. 5* (1986), 391-421.)

[Sh2] —— Singularities of families of involvents in the neighbourhood of an inflection point of the curve, and the group H3 generated by reflections. *Functional Anal. Appl. 17* (1983), 301-303.

[Sh3] —— Wavefronts and reflection groups. *Russian Math. Surveys 43, 3* (1988), 149-194.

[VC] Varchenko, A.N. and Chmutov, S.V. Finite irreducible groups, generated by reflections, are monodromy groups of suitable singularities. *Functional Anal. Appl. 18* (1984), 171-183.

[Va] Vassilyev, V.A. *Lagrange and Legendre characteristic classes.* Gordon and Breach. 1988.

(Reçu le 6 avril 1989)

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