§1. Basic definitions

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CONTACT GEOMETRY AND WAVE PROPAGATION [1)]

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Dedicated to the memory of O. P. Shcherbak

§ 1. BASIC DEFINITIONS

Symplectic geometry is at present generally accepted as the natural basis for mechanics and for the calculus of variations.

Contact geometry, which is the odd-dimensional counterpart of the symplectic one, is not yet so popular, although it is the natural basis for optics and for the theory of wave propagation.

The relations between symplectic and contact geometries are similar to those between linear algebra and projective geometry. First, the two related theories are formally more or less equivalent: every theorem in symplectic geometry may be formulated as a contact geometry theorem, and any assertion in contact geometry may be translated into the language of symplectic geometry. Next, all the calculations look algebraically simpler in the symplectic case, but geometrically things are usually better understood when translated into the language of contact geometry.

Hence one is advised to calculate symplectically but to think rather in contact geometry terms.

Finally, most of the global, topological results are more natural in the contact gometry context, and so we can up-date the well known slogan "projective geometry is all geometry" by saying "contact geometry is all geometry". For instance, most of the facts of the differential geometry of submanifolds of Euclidean or of Riemannian space may be translated into the language of contact geometry and may be proved in this more general setting. Thus we can use the intuition of Euclidean or Riemannian geometry to guess the general results of contact geometry, whose applications to the problem of

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ordinary differential geometry provide new information in this classical domain.

Definition. A *contact structure* on an odd-dimensional manifold is a field of tangent hyperplanes, which is generic at every point.

These hyperplanes are called the *contact hyperplanes* (of a given contact structure).

By a theorem of Darboux (that we shall prove later) all the generic fields of tangent hyperplanes are locally diffeomorphic to each other. Hence one may consider one particular example of a contact structure on a manifold of a given dimension, and substitute the genericity condition by the condition of being equivalent to the chosen normal form.

Such a normal form may be described as the field of hyperplanes given by $\alpha = 0$, where

$$\alpha = dy - pdq \tag{1}$$

(here $y, p = (p_1, \ldots, p_n)$, $q = (q_1, \ldots, q_n)$ are called the Darboux coordinates).

The nondegeneracy condition for the field $\alpha = 0$ is $\alpha \wedge (d\alpha)^n = 0$. Or, equivalently, $d\alpha$ defines a bilinear symplectic structure on each contact hyperplane $\alpha = 0$. Instead of the normal form (1) one uses also such forms as

$$\alpha = dz + (pdq - qdp)/2\,. \tag{2}$$

These coordinates are also called Darboux coordinates (sometimes the 2 is dropped or some of the signs are inversed).

Example 1. Let us consider the set of 1-jets of functions $f\colon V^n \to \mathbf{R}$.

Let $(q_1, \ldots, q_n)$ be local coordinates on $V = V^n$ and y be the coordinate in $\mathbf{R}$. A 1-jet of a function $y = f(q)$ is defined by the value of the function and of its first partial derivatives at a given point. Hence the manifold of all 1-jets $J^1(V, \mathbf{R})$ is of dimension $2n + 1$, and its local coordinates are $(p_1, \ldots, p_n;\ q_1, \ldots, q_n; y)$, where $p_k = \partial f/\partial q_k$ (in PDE the coordinates (p, q, y) are usually denoted by (ξ, x, u)).

The manifold of 1-jets is equipped with a *natural contact structure* defined locally by the equation

$$dy = pdq\,.$$

This contact structure is independent of the particular choice of the coordinates and hence is defined globally. To see this, let us associate to any function f

its 1-graph, which is the set of its 1-jets at all the points of V. The tangent spaces to the 1-graphs of all the functions at any given point of $J^1(V, \mathbf{R})$ belong to a hyperplane in the tangent space to J^1 at that point. This hyperplane's local equation is $dz = pdq$. Hence this equation defines a hyperplane independent of any coordinate system: it admits an intrinsic definition, given above.

This contact structure of the manifold of 1-jets is called its canonical (or natural) contact structure.

Example 2. A *contact element* on a given manifold is a hyperplane in a tangent space. The set of all contact elements on a given manifold $B = B^m$ is fibered over B, and the fibre over a point of B is the projectivized cotangent space of B at this point (called the point of contact).

This set of all contact elements of B is called the space of the *projectivized cotangent bundle* PT^*B. Its dimension $2m - 1$ is odd and it carries a natural contact structure.

This structure is defined by the following construction. A velocity of motion of a contact element is called admissible if the velocity of the point of contact belongs to the contact element. It is easy to see that the admissible velocities form a hyperplane at any given point of PT^*B, and that these hyperplanes define a contact structure.

The set of all the contact elements, tangent to any particular submanifold of B, is an integral manifold of this contact structure of PT^*B. The dimension of such integral manifolds is independent of the dimensions of the original submanifold: it is always $m - 1$, that is almost one half of the dimension of the whole contact manifold.

The integral submanifolds of this maximal dimension of a contact structure are called *Legendre manifolds.*

Thus to every submanifold of the base manifold B there corresponds a Legendre submanifold of the contact elements' manifold PT^*B. For instance, let us start with a point of B (a 0-dimensional submanifold). The corresponding Legendre manifold is the fibre of the bundle $PT^*B \to B$. Hence these fibres are Legendre submanifolds. A fibration (or a foliation) of a contact manifold whose fibres (leaves) are Legendre submanifolds is called a Legendre fibration (foliation). Thus the projectivized cotangent bundle is a Legendre fibration. Another example is the natural fibration $J^1(V, \mathbf{R}) \to J^0(V, \mathbf{R})$ which is the "forgetting of the derivatives" mapping.

At the other extremity we have the hypersurfaces of B. In this case the Legendre submanifold consists of the tangent spaces of the hypersurface. It is naturally diffeomorphic to the hypersurface.

Example 3. Let us consider the particular case PT^*P^n (the base space is the projective space). In this particular case there exists a natural isomorphism

$$PT^*P^n \approx PT^*(P^{n*})$$

where P^{n*} is the dual projective space of P^n. This isomorphism associates to a contact element of P^n (that is to a projective hyperplane and to one of its points) the dual contact element of P^{n*} (consisting of the projective hyperplane considered as a point of P^{n*}, and of the point of P^n, considered as a hyperplane in P^{n*}).

Thus our manifold of contact elements of the projective space is equipped with two contact structures: the first is that of PT^*P^n, the other comes from PT^*P^{n*}.

These two contact structures coincide. (This is a non-trivial theorem having an easy geometrical proof; I leave to the reader the pleasure of discovering it.)

Now let us consider a smooth hypersurface in P^n. Its tangent contact elements form a Legendre submanifold L in PT^*P^n. The fibration $PT^*P^n \to P^{n*}$ is a Legendre fibration. The image of our Legendre submanifold L in P^{n*} is called the *dual hypersurface* of the initial one.

We see that the dual hypersurface of a smooth hypersurface is the image of the corresponding Legendre submanifold of PT^*P^n under the Legendre projection $PT^*P^n \to P^{n*}$.

The image of a Legendre submanifold under a Legendre projection is called the *front* (of the corresponding Legendre submanifold). Hence the dual hypersurface is the front of the Legendre submanifold of tangent hyperplanes of the initial hypersurface.

The affine version of this projective construction is called the Legendre transformation. More precisely, if the initial hypersurface is given by the equation $z = f(q)$, and the dual one by $w = g(p)$, then the function g is called the Legendre transformation of f.

The triple $L \to E \to B$ where the left arrow is the embedding of a Legendre submanifold and the second is a Legendre fibration is called a Legendre projection. A germ of a Legendre projection at a point is called a Legendre singularity.

An equivalence of two Legendre projections (or of two Legendre singularities) is a commutative diagram

$$\begin{array}{ccccc} L_1 & \to & E_1 & \to & B_1 \\ \downarrow & & \downarrow & & \downarrow \\ L_2 & \to & E_2 & \to & B_2 \end{array}$$

whose vertical maps are diffeomorphisms, such that the middle vertical map preserves the contact structure.

Equivalent Legendre projections (or singularities) have diffeomorphic fronts.

Example 4. Let us consider a hypersurface in a Riemannian manifold. The equidistant hypersurfaces are the fronts of the appropriate Legendre mappings.

To see this let us consider the geodesic flow of the oriented contact elements. At time t the element contact point moves at distance t along the geodesic, orthogonal to the element, in the direction defined by the co-orientation. And the moving contact element is always orthogonal to the geodesic.

The geodesic flow of contact elements of B is a one-parameter family of diffeomorphisms of the manifold ST^*B of co-oriented contact elements.

THEOREM. *The geodesic flow of contact elements preserves the natural contact structure of the manifold of contact elements.*

This non-trivial geometrical theorem is one of the formulations of Huygens' principle, which describes the moving wavefront as the envelope of the spherical fronts issuing from the points of the initial front.

The diffeomorphisms of a contact manifold preserving contact structures are called contactomorphisms. They form the contactomorphism group of the manifold — one of the (pseudo) groups of E. Cartan's list of series of simple infinite dimensional groups (the other series are the diffeomorphism groups, the symplectomorphism groups, the biholomorphism group and so on).

Remark. Locally any contact structure is defined as the field of zeroes of an 1-form; any such form is called a *contact form* (associated to a given contact structure). Sometimes one can choose such a form globally (this is the case, for instance, for the 1-jet space's natural contact structure).

In such cases one is tempted to consider the group of diffeomorphisms, preserving the contact form (which is only a subgroup of the true group of contactomorphisms). Some bad people require contact transformations to be elements of this subgroup. This is a mistake to avoid: the subgroup is not intrinsically related to the contact structure, it depends on the particular choice of the contact form.

Example 5. The *pedal hypersurface* of a given hypersurface in the Euclidean space is formed by the points where the tangent hyperplanes meet the perpendiculars issuing from the origin.

The pedal hypersurfaces of smooth hypersurfaces have singularities. Some experimentation shows that they are (generically) the same as singularities of (generic) equidistants or that of hypersurfaces dual to the (generic) smooth ones of the fronts of the generic Legendre submanifolds.

For instance, in the case of plane curves the generic front singularities are just cusp points of order 3/2 (the front being locally diffeomorphic to the semicubical parabola $x^2 = y^3$). The generic singularities of the front surfaces in the usual 3 space are cusped edges and swallow tails — the surfaces, locally diffeomorphic to the set of polynomials $x^4 + ax^2 + bx + c$, having multiple (real) roots (this swallowtail hypersurface has been studied by Kronecker).

Example 6. Let us consider a smooth hypersurface in a Euclidean space (the origin excluded). The family of all the hyperplanes, normal to the radius-vectors of the hypersurface of all the points of the hypersurface has an envelope. This envelope may have singularities. Some experimentation shows that they are the same as those of the fronts, equidistants of the graphs of the Legendre transformations and so on, as in the preceeding example.

The geometric reason for these coincidences is that in all these different cases, there is somewhere a Legendre singularity.

THEOREM. *The transformations leading to the singular hypersurfaces of examples 5 and 6 may be described as products of the projective duality transformation and the inversion transformation.* (The order of the use of these two involutions in the two examples is different.)

Since the inversion transformation is a diffeomorphism, the singularities of examples 5 and 6 are Legendre singularities.

Example 7. Let $V = V^{2n}$ be a vector space of dimension $2n$, equipped with a linear symplectic structure (a nondegenerate bilinear skew-symmetrical form). The projectivized space P^{2n-1} carries a natural contact structure. (We associate to a point of V its skew-orthocomplement hyperplane, thus defining a hyperplane field on P^{2n-1}.)

This contact structure is invariant under the natural action of the linear symplectic group on P^{2n-1}.

Example 8. Let $V = \{a_0x^d + \ldots + a_dy^d\}$ be the vector space of all binary forms of degree d. The group SL_2 of linear unimodular transformations of the (x, y)-plane acts on V. If d is odd, the dimension of V which is $d+1$ is even and V carries a linear symplectic structure, invariant under the SL_2 action. This 2-form is unique up to a nonzero multiple. It defines on V a translation invariant symplectic structure.

[The explicit formula for this structure may be described in terms of the Darboux coordinates reducing the symplectic form to $\Sigma dp_i \wedge dq_i$. The expression of the binary form $\phi \in V$ in terms of these Darboux coordinates is

$$\phi(x, 1) = q_1 X_d + \dots + q_n X_n - p_n X_{n-1} + \dots + (-1)^n p_1$$
$$\text{where} \quad X_j = x^j/j!\,, \quad d = 2n - 1$$

and the signs preceeding the p_m's alternate.]

Combining this natural symplectic structure of the space of binary forms with the construction of example 7 above we obtain:

PROPOSITION. *The projective space of the 0-dimensional hypersurfaces of degree $d = 2n - 1$ of the projective line carries a natural $PL(2)$-invariant contact structure.*

[For instance, in the domain of the projective space of hypersurfaces represented by the polynomials $\phi(x, 1)$ with $q_1 = 1$, the above contact structure is defined as the set of zeroes of the 1-form

$$\alpha = p'dq' - q'dp' - dp_1$$
$$p' = (p_2, \dots, p_n)\,, \quad q' = (q_2, \dots, q_n)] \,.$$

COROLLARY. *The set of 0-dimensional hypersurfaces of degree $2n - 1$ containing any given point with multiplicity n (or higher than n) is a Legendre submanifold.*

This follows from the above explicit formula for the case of the point $x = 0$. Since the contact structure is PL_2 invariant, it is still true for any given point.

Example 8. The Gibbs (1873) "graphical methods in thermodynamics" is pure contact geometry. Let ν be the volume, p the pressure, t the (absolute) temperature, ε the energy, η the entropy. The contact structure of thermodynamics is defined by the equation

$$d\varepsilon = t d\eta - p d\nu \,.$$

Every substance is represented as a Legendre 2-surface in this 5-dimensional contact manifold of the thermodynamical states. Different physical states of the same substance correspond to different points of this Legendre surface, but the evolution of state is severely restricted by the contact structure.

§2. Characteristics

Let us consider a hypersurface in a contact manifold.

Example 1. A hypersurface in the manifold of 1-jets of functions is called a 1st order nonlinear differential equation.

Example 2. Let us consider pseudo-Euclidean space-time. Among all the contact elements of the space-time one distinguishes the light elements: those tangent to the light cone. The light elements form a hypersurface in the contact space of all the contact elements of the space-time.

More generally, a hyperbolic PDE (or a hyperbolic system of PDE) defines a field of "Fresnel cones" (of cones of zeroes of the principal symbol) in the cotangent spaces of the space-time manifold. This field defines the "light hypersurface" in the contact space of all the contact elements of the space-time manifold. The contact geometry of this hypersurface is crucial for the understanding of the propagation of the waves defined by the hyperbolic equation (or system).

The tangent plane to a hypersurface in a contact manifold is generically different from the contact plane (they coincide, generically, at some isolated singular points of the hypersurface). Let us first consider the nonsingular points. At a nonsingular point the hypersurface in the contact manifold carries a distinguished tangent line, called the characteristic direction. The word "characteristic" in mathematics always means "intrinsically associated". Thus the characteristic equation of a matrix of an operator is independent of the basis, the characteristic subgroups are invariant under automorphisms, and so on.

The *characteristic direction* at a given point of a hypersurface in a contact manifold may be defined as the only direction associated intrinsically to the hypersurface and to the contact structure. Indeed, the subgroup of the contactomorphisms which preserve the point and the hypersurface, preserves exactly one line tangent to the hypersurface at this point. (We still suppose that the