§2. Characteristics

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Every substance is represented as a Legendre 2-surface in this 5-dimensional contact manifold of the thermodynamical states. Different physical states of the same substance correspond to different points of this Legendre surface, but the evolution of state is severely restricted by the contact structure.

§2. CHARACTERISTICS

Let us consider a hypersurface in a contact manifold.

Example 1. A hypersurface in the manifold of 1-jets of functions is called a 1st order nonlinear differential equation.

Example 2. Let us consider pseudo-Euclidean space-time. Among all the contact elements of the space-time one distinguishes the light elements: those tangent to the light cone. The light elements form a hypersurface in the contact space of all the contact elements of the space-time.

More generally, a hyperbolic PDE (or a hyperbolic system of PDE) defines a field of "Fresnel cones" (of cones of zeroes of the principal symbol) in the cotangent spaces of the space-time manifold. This field defines the "light hypersurface" in the contact space of all the contact elements of the space-time manifold. The contact geometry of this hypersurface is crucial for the understanding of the propagation of the waves defined by the hyperbolic equation (or system).

The tangent plane to a hypersurface in a contact manifold is generically different from the contact plane (they coincide, generically, at some isolated singular points of the hypersurface). Let us first consider the nonsingular points. At a nonsingular point the hypersurface in the contact manifold carries a distinguished tangent line, called the characteristic direction. The word "characteristic" in mathematics always means "intrinsically associated". Thus the characteristic equation of a matrix of an operator is independent of the basis, the characteristic subgroups are invariant under automorphisms, and so on.

The *characteristic direction* at a given point of a hypersurface in a contact manifold may be defined as the only direction associated intrinsically to the hypersurface and to the contact structure. Indeed, the subgroup of the contactomorphisms which preserve the point and the hypersurface, preserves exactly one line tangent to the hypersurface at this point. (We still suppose that the tangent plane of the hypersurface is transversal to the hyperplane of the contact structure at this point.)

One might prefer a more explicit definition of the characteristic direction. For the simplest case of a 3-dimensional contact manifold it is the intersection of the contact plane with the tangent plane of the hypersurface at the given point.

In the general case the intersection is not a line, but a hyperplane in the contact hyperplane. To transform a hyperplane into a line, we use the natural (conformally) symplectic structure of the contact hyperplane.

Let α be a contact form, defining (locally) our contact structure, $\alpha = 0$. The 2-form $d\alpha$ defines the bilinear symplectic structures in the contact hyperplanes $\alpha = 0$. $d\alpha$ is nondegenerate at the plane $\alpha = 0$, since α is a contact form. The 1-form α is defined up to multiplication by nonzero functions. Since the 2-form $d(f\alpha) = df \wedge \alpha + f d\alpha$ is reduced to $f d\alpha$ at the plane $\alpha = 0$, the bilinear symplectic form $d\alpha$ in the plane $\alpha = 0$ is well defined up to a nonzero constant multiple.

The characteristic direction at a point of a hypersurface in a contact manifold is the skew-orthocomplement to the intersection of the tangent plane of the hypersurface with the contact plane at the given point. The characteristic direction of a hypersurface is tangent to the hypersurface, since the skew-orthocomplement of a hyperplane in a symplectic space belongs to the hyperplane.

The integral lines of the field of characteristic directions are called the characteristics of the given hypersurface.

Example. Let us consider the hypersurface $K = 0$ in the space with contact structure $\alpha = 0$, where

$$\alpha = dz + \frac{pdq - qdp}{2}.$$

[After some calculations] one finds the equation of the characteristics:

$$\dot{p} = -K_q + \frac{p}{2}K_z, \quad \dot{q} = K_p + \frac{q}{2}K_z, \quad \dot{z} = -\frac{p}{2}K_p - \frac{q}{2}K_q.$$

Example. The characteristics of the light hypersurface in the projectivized cotangent bundle of the space-time manifold defines the wave propagation for the corresponding (pseudo) differential hyperbolic equation. This is one more form of the Huygens principle.

A submanifold of a contact manifold, tangent to the contact planes, is called an *integral submanifold* of the contact structure. For instance, a Legendre submanifold is a maximal integral manifold.

Let us consider an integral submanifold of the contact structure, lying in a given hypersurface of the ambient contact manifold. Let us suppose that the characteristics of the hypersurface are nowhere tangent to the submanifold. An easy fact on characteristics is the following

THEOREM. *The characteristics, issuing from points of the given integral submanifold, form (at least locally) an integral submanifold.*

COROLLARY 1. *The characteristics of a Legendre submanifold belong to the submanifold.*

This property of the characteristics can be used as their definition.

COROLLARY 2. *If the dimension of the integral submanifold in the theorem is one less than the dimension of a Legendre manifold, then the characteristics intersecting the submanifold form (at least locally) a Legendre submanifold — the unique Legendre submanifold of the hypersurface containing the given integral submanifold.*

This corollary contains the theory of the Cauchy problem for first order nonlinear PDEs. Such a PDE $F(p, q, y) = 0$ defines a hypersurface in the space of 1-jets of functions $y(q)$, equipped with its natural contact structure $dy = pdq$. The initial data define an integral submanifold. The characteristic's equation is

$$\dot{q} = F_p, \quad \dot{p} = -F_q - pF_y, \quad \dot{y} = pF_p.$$

The Legendre submanifold, formed by the characteristics, is the 1-graph of the solution.

As another example let us consider a hypersurface in space. The contact elements of space-time, tangent to this submanifold and belonging to the light hypersurface, form an integral submanifold of the contact structure. The characteristics of the light hypersurface, issuing from points of this submanifold, form a Legendre submanifold of the light hypersurface in the space of contact elements of the space-time which is called the *big front*. The intersections of the big front with the isochrones are called the *momentary fronts*.

Thus the light hypersurface of the projectivized cotangent bundle of the space-time (considered as a hypersurface in a contact manifold) defines both the evolution of the wave fronts (the Legendre manifold) and the rays (the characteristics). This description of the wave propagation is one more formulation of Huygens' principle.

There exist two other types of characteristic in a contact manifold. Let us consider a contact form α on a contact manifold (a form, defining its contact structure, $\alpha = 0$). Such a form might be nonexistent globally and it is not intrinsically defined by the contact structure, but we can choose it locally.

The 2-form $d\alpha$ is as nondegenerate as possible for a skew-symmetric 2-form in an odd-dimensional space — it has at every point a 1-dimensional kernel. This kernel direction is called the *characteristic direction of the contact 1-form* α. It is the only direction, associated to the 1-form intrinsically (i.e. preserved by the contactomorphisms which do not change the form).

The integral lines of this field of directions are called the characteristics of the 1-form. They are transversal to the contact planes.

Example 1. The characteristics of the form $dz - pdq$ are the vertical lines generated by the vector field $\partial/\partial z$. The same is true for the form $dz + \frac{pdq - qdp}{2}$.

Example 2. Let us consider the standard embedding of S^3 in $\mathbf{C}^2$ or in the standard symplectic $\mathbf{R}^4$. The complex structure of $\mathbf{C}^2$ defines on S^3 a field of complex lines, which is a contact structure.

The symplectic structure of $\mathbf{R}^4$ also defines on S^3 a field of 2-planes (consider the skew-orthocomplements to the radius-vectors). This 2-plane field coincides with the contact structure induced on S^3 by the complex structure of $\mathbf{C}^2$ and is called the *natural* (or *standard*) *contact structure* of S^3.

This structure may be defined globally as the field of zeroes of an $SU(2)$-invariant 1-form. This 1-form is unique up to a scalar multiple.

The characteristics of this 1-form are some of the great circles of the sphere. They are the fibres of the Hopf fibration $S^3 \to S^2$.

In both these examples the characteristics of a contact form form a symplectic manifold. Indeed, the 2-form $d\alpha$ is well defined on the space of characteristics, since it vanishes along the characteristics.

COROLLARY. *Let us consider any Legendre curve (that is, any integral curve) of the contact structure in* $\mathbf{R}^3$ *defined by the contact form* $dz - pdq$.

Then there exists at least one characteristic chord of this curve (that is, at least one characteristic of the form intersecting the curve twice).

Proof. Let us consider the projection of the curve along the characteristics to the space of the characteristics (that is the projection to the $(p, q$-plane) along the vertical z-lines). Since the curve was a Legendre curve, its projection is a 0-area curve ($\oint p\,dq = 0$). Hence the projection has self-intersections, and the Legendre curve has characteristic chords.

An old conjecture on the contact 1-forms defining the standard contact structure on S^3, is the existence of characteristic chords for any Legendre curve (warning: a characteristic chord may have geometrically only one common point with the Legendre curve if this chord is closed, i.e. diffeomorphic to a circle).

This is a particular case of a set of general conjectures in the higher-dimensional contact topology of Legendre manifolds of greater dimension. These conjectures are not discussed here since even the simplest case of Legendre curves in S^3 still remains unsettled.

The two-sided relation between the contact and symplectic geometries depends on the fact that one can obtain an even number from an odd one either by adding or by subtracting one.

THEOREM. *The manifold of the characteristics of a contact form has a natural symplectic structure.*

Example. Let us consider the unit sphere $S^{2n-1} \subset \mathbf{C}^n$ with its natural contact form. The space of characteristics is $\mathbf{CP}^n$ with its natural symplectic form. Let us consider a contact structure, defined globally by a contact form.

THEOREM. *The total space of the bundle of all the linear forms on the tangent spaces to a contact manifold, which are zero exactly on the contact hyperplanes, has a natural symplectic structure.*

Indeed, this manifold is fibred over the contact manifold (with fibres $\mathbf{R}\backslash 0$). Each point of the bundle is a linear form on the tangent space to the total space at our point. This way we define a canonical 1-form α on the total space. The required symplectic structure is $d\alpha$.

The symplectic manifold thus obtained is called the symplectization of the original contact manifold.

Conversely, one may start from a symplectic manifold and obtain (at least locally) a contact manifold whose dimension is greater by 1 or smaller by 1 than that of the symplectic manifold.

The two standard models for this are the fibrations

$$\begin{array}{ccc} J^1(V,\mathbf{R}) & \to & T^*V \\ \text{contact} & & \text{symplectic} \\ dy - pdq & & dp \wedge dq \end{array}$$

and

$$\begin{array}{ccc} T^*B\backslash B & \to & \mathbf{P}T^*B \\ \text{symplectic} & & \text{contact} \\ dp \wedge dq & & pdq = 0\,. \end{array}$$

We may think of $J^1(V,\mathbf{R})$ as being the *contactization* of T^*V and of $T^*B\backslash B$ as being the *symplectization* of PT^*B.

Example. Let L be an (immersed) Legendre submanifold of $J^1(V,\mathbf{R})$. Then its projection to T^*V is an (immersed) Lagrange submanifold of T^*V.

Let us start with an arbitrary connected Lagrange submanifold Λ in T^*V. Let us fix a point in $J^1(V,\mathbf{R})$ over some point of Λ. Then locally there exists one and only one Legendre manifold in $J^1(V,\mathbf{R})$ containing the fixed point and projecting diffeomorphically to Λ ($y = \oint pdq$ is locally independent of the path joining two given points).

But globally such a Legendre manifold might exist or not exist. In the first case the initial Lagrange manifold Λ is called an *exact Lagrange manifold* (since in this case the 1-form pdq is exact on Λ).

For instance, the graph of the differential of a function on V is an exact Lagrange submanifold of T^*V.

Definition. A quasifunction on V is an imbedded Legendre submanifold of $J^1(V,\mathbf{R})$, which is isotopic to the zero section in the class of the imbedded Legendre submanifolds.

A quasicritical point of a quasifunction is a point where its Lagrange projection intersects the zero section of T^*V.

THEOREM (Tchekanov). *The number of quasicritical points of a quasi function on a compact manifold is bounded below by the sum of the Betti* *numbers of the manifold (when counted with multiplicities) and the number of geometrically different critical points is bounded below by the cuplength of the manifold.*

One conjectures that both the "algebraic" and the "geometric" numbers are bounded below by the minimal "algebraic" and "geometric" numbers of critical points of a function on the manifold. But this is proved not for the functions on the original manifold, but for functions on a vector bundle over the manifold coinciding with a nondegenerate quadratic form of signature zero at the infinity of every fibre.

The characteristics of the third type are the orbits of the flows of contactomorphisms. Let us suppose that the contact structure is defined by a global 1-form α and let us fix this 1-form.

A vector field on a contact manifold is called a *contact vector field,* if its flow preserves the contact structure. The contact vector fields form a Lie algebra — the Lie algebra of contact vector fields.

Let V be a contact vector field and let α be the fixed contact form. Then the function $K = \alpha | V$ is well defined. This function is called the *contact Hamilton function.* If we choose another contact form the contact Hamilton function acquires a nowhere zero functional multiplier. Hence the hypersurface (the "divisor") of the zeroes of K is intrinsically defined by the contact structure and by the contact vector field. The choice of a global contact form is a trivialization of the line bundle of the linear forms vanishing on the contact planes. The contact Hamilton function K is in fact a section of the dual line bundle. But I prefer to fix a trivialization and to call K a function.

THEOREM. *Any function on a contact manifold is the contact Hamilton function of some contact vector field, which is defined uniquely by this function.*

In Darboux coordinates, where

$$\alpha = dz + \frac{pdq - qdp}{2}$$

the contact vector field V with the contact Hamilton function K defines the "contact Hamilton differential equations"

$$\dot{p} = -K_q + \frac{p}{2}, \quad \dot{q} = K_p + \frac{q}{2}K_z, \quad \dot{z} = K_p + \frac{q}{2}K_z.$$

Since the last part of the theory is computational, it might be easier to understand it from the symplectic geometry point of view.

Let us consider the manifold of 1-forms in the tangent spaces to our contact manifold which vanish exactly on the contact planes.

This manifold has a natural symplectic structure (defined above) and a natural action of the multiplicative group **R*** of nonzero scalars. It is a principal **R*** bundle over the initial contact manifold.

We lift the contactomorphisms to the bundle space and we obtain the symplectomorphisms, commuting with the action of the multiplicative group. The corresponding Hamilton functions may be chosen to be the homogeneous functions of degree 1 along the fibres.

The choice of the bundle's trivialization α associates to these homogeneous functions contact Hamilton functions. The formula for the contact vector field is the usual formula for the Hamilton field, projected to the contact base from the symplectic total space (taking into account the homogeneity).

In the same way, the usual Poisson bracket on the symplectic **R*** bundle space, restricted to the homogeneous functions of degree 1 along the fibres, defines a "Lie bracket" on the space of functions on a contact manifold. This contact Lie bracket is a particular case of a "Lie structure" on a manifold: it is a Lie algebra structure on the space of functions, the bracket being a first order differential operator in each of its two arguments.

Lie structures are generalizations of *Poisson structures* (which correspond to brackets homogeneous in the derivatives). A Poisson manifold is decomposed into a collection of *symplectic leaves* (manifolds of different dimensions) unified by the smoothness of the common Poisson bracket. The simplest example is the Lie-Berezin-Kostant-Kirillov bracket on the coadjoint space of a Lie algebra. For the $SO(3)$ case the leaves are defined by the equation $M_2 = \text{const}$ (they are spheres of dimension 2 when the constant is positive, and their common centre — a symplectic manifold of dimension 0 — when the constant vanishes).

The leaves are defined as the sets of points, attainable from a given one by Hamiltonian paths, [namely a Poisson structure on a manifold associates to a function on the manifold a vector field such that the Poisson bracket with this function is differentiation along this field. This field is called the *Hamiltonian field,* associated to a given Hamilton function. The Hamiltonian paths are defined by the time-dependent Hamilton functions (or, equivalently, by broken paths containing several segments of the time-independent Hamiltonian field's orbits).]

For Poisson structures there exists an elaborate theory of singularities, mainly reducing them locally to the transversal slice of the singular leaf

(A. Weinstein). In coordinates the Poisson structure has the form $\{a, b\} = \Sigma c^i_{jk}(x)\partial a/\partial x_j \partial b/\partial x_k$. For the transversal structure all the coefficients c vanish at the origin.

The linear part of the transversal structure defines a finite dimensional Lie algebra. If this algebra is semisimple, the whole transversal structure is equivalent to its linear part ([Co]). To be precise, the last theorem holds for analytic Poisson structures. In the C^∞ case the semisimple algebra should be the Lie algebra of a compact Lie group.

The problem of the linearization of transverse Poisson structures was discussed in my 1963 paper [A1]. The reduction of the 3-body problem here is based on the study of the Poisson leaves for $O(3)$.

Interesting Poisson structures are provided by the period mappings of the versal deformations of singular hypersurfaces ([GV]). The simplest of these Poisson structures lives in the three-dimensional (a, b, c) space of the swallowtail. [It is described axiomatically by the 3 conditions: (i) all the leaves are two-dimensional, (ii) the self-intersection line of the swallowtail is contained in one of the leaves, and (iii) the leaf containing the origin is transversal to the tangent plane of the swallowtail at the origin. For the swallowtail $\{x^4 + ax^2 + bx + c = (x+t)^2 \ldots\}$ the tangent plane is $dc = 0$.]

One can locally reduce any such structure to the normal form $\{a, c\} = 1$, $\{a, b\} = \{b, c\} = 0$ by a local swallowtail-preserving diffeomorphism. The leaves are the vertical planes $b = \text{const}$. The Poisson structure's normal form implies this normal form of the corresponding fibration of the space into planes.

A Lie structure on a manifold, like a Poisson structure, defines a decomposition of the manifold into submanifold leaves. In this case some of the leaves are even dimensional symplectic manifolds, while others are odd dimensional contact manifolds ([Ki]).

The transversal structure theory for this case has not been yet constructed, as far as I know.

From the algebraic point of view Poisson structures are better objects than honest symplectic manifolds, and general Lie structures are better than nondegenerate contact ones.

The algebraic objects include automatically all the degenerations of the corresponding geometric structures. Perhaps because of this advantage the algebraic theory is so difficult that it contains almost no general results. The few general results that I know were first obtained geometrically (for some mild degenerations) and then the geometrical proofs were translated into the algebraic language, and hence had become general. The dictionary of the translation is simple: one substitutes principal ideals for the hypersurfaces, one controls the Poisson brackets vanishing at subvarieties by corresponding conditions on the ideals and so on. For instance, the nondegeneracy condition for the hypersurface $f = 0$ is "$\{f, g\} = 1$ for some function g". The algebraic study of the degenerate symplectic and contact varieties is an important but almost unexplored domain.

For instance, let us consider the generalized swallowtail surface $\{x^5 + Ax^3 + Bx^2 + Cx + D = (x+t)^2 \ldots\}$ in the 4-space of the versal deformation of the singularity A_4 (i.e. x^5 or $SU(5)$). The Givental-Varchenko period mapping equips this 4-space with a natural symplectic structure (one may find an explicit formula in the last chapter of the book [AGV]). The generalized swallowtail has with respect to this structure very special properties (for instance, the selfintersection subvariety $\{(x+t)^2(x+s)^2 \ldots\}$ is Lagrangian, because different cycles, vanishing at the same critical level of a function, do not intersect each other. But it is unkown whether this swallowtail is uniquely defined (up to symplectomorphisms) by the ranks of the restrictions of the symplectic form to the tangent cones of its strata.

Let us return to a usual contact space, equipped with its Darboux coordinates (p, q, z) and with its Darboux contact form $\alpha = dz + \frac{pdq - qdp}{2}$. Let K, as above, be the contact Hamilton function of a contact vector field V. We shall denote by a dot over the name of a function the derivative of this function along V. The explicit formula for the components of the field V in the Darboux coordinates is given above (it precedes the Lie structure discussion). This formula implies the

COROLLARY. $\dot{K} = KK_z$, $L_V\alpha = K_z\alpha$ *(where* L *is the Lie derivative). If* $H(p, q)$ *is a quadratic form, then*

$$\dot{H} = \{K, H\} + HK_z$$

where $\{K, H\} = K_pH_q - H_pK_q$ *is the symplectic Poisson bracket.*

Now we shall compare the three types of characteristics.

Let us fix a contact 1-form α and a function K. Let us consider all the three characteristic directions at every point: that of the hypersurface $K = \text{const}$, that of the 1-form α and that of the contact Hamilton vector field, defined by K.

In general they are different.

THEOREM. *These three lines lie in a 2-plane.*

Proof. This is clear from the above formula for the vector field V in Darboux coordinates and from the formula for the characteristics of a hypersurface. The only difference is the additional term $K\partial/\partial z$ in the formula for V. This additional term is vertical, i.e. directed along the characteristics of the 1-form α.

Exercise. Find a direct proof, independent of the Darboux coordinates.

The tangent hyperplanes of a generic hypersurface in a contact manifold coincide with the contact planes at some isolated points of the hypersurface. Those points are called the characteristic points. They are the singular points of the field of the characteristic directions.

A smooth generic hypersurface may be reduced to a simple normal form in a neighbourhood of its characteristic points by a smooth contactomorphism (Lychagin, 1975). In Darboux coordinates the Lychagin normal form is

$$z = Q(p, q),$$

where Q is a nondegenerate quadratic form. In the complex case one may reduce Q further to the "eigenvalues" normal form $Q = \Sigma\lambda_k p_k q_k$ (since any linear symplectomorphism, $(p, q) \mapsto (\tilde{p}, \tilde{q})$ induces a contactomorphism $(p, q, z) \mapsto (\tilde{p}, \tilde{q}, z)$).

The classification of characteristic points of holomorphic hypersurfaces in contact manifolds is very different from this simple normal form. The formal series, which reduces the hypersurface and the contact structure to their normal forms is generically divergent.

However in the case of a 3-dimensional contact manifold the situation is better. Let us consider an implicit ordinary differential equation defined by the hypersurface $F(p, q, y) = 0$ in the 3-space of 1-jets of functions $y = f(q)$ equipped with its natural contact structure $dy = pdq$ and with the structure of the Legendre fibre bundle $J^1 \to J^0((p, q, y) \mapsto (q, y))$.

The characteristics of the surface $F = 0$ projected to the base (q, y)-plane, are called "integral curves" of the implicit differential equation.

For a generic function F the surface $F = 0$ is smooth, but its projection to the base plane may have singularities. Those singularities are generically fold lines and ordinary Whitney cusp points.

The contact planes are vertical (they contain the fibre directions). Hence the characteristic points of the surface belong to the fold line (generically those points will not coincide with the cusp points).

THEOREM (A. Davydov, 1985). *In a neighbourhood of a characteristic point of the surface* $F = 0$ *a generic implicit differential equation may be reduced to the local normal form*

$$y = (p + kq)^2 \qquad (*)$$

by a local diffeomorphism of the base (y, q)*-plane.*

This diffeomorphism is C^∞ for the C^∞-equations, analytic for the analytic equations and so on. The (real) number k is a modulus — a parameter, on which the normal form depends.

A (local) diffeomorphism of the base space of our Legendre fibration induces a (local) contactomorphism, of the fibration total space. Hence the surface, $F = 0$ is reduced to the special form (*) by a (local) contactomorphism. Thus the family of characteristics on a generic surface in a contact 3-space is locally diffeomorphic (in a neighbourhood of a characteristic point) to the family of the characteristics on the standard surface (*).

The differential equation (*) is easy to solve and the corresponding phase curves on the surface form (for any generic value of k) one of the standard Poincaré patterns: the node, the focus, the saddle.

The folding mapping defines an involution on the surface (interchanging the two counterimages of any base point). The essential part of Davydov's proof is the proof of the equivalence of all the generic involutions, occurring in such a situation. Namely one considers the involutions of the plane, whose curves of fixed points contain the singular point of a vector field, this field being anti-invariant under the involution at the line of fixed points of the involution. Those two properties of the involution and the genericity are sufficient for the stability of the involution with respect to a generic vector field: all the neighbouring involutions with the same properties may be transformed into a given one by a diffeomorphism which preserves orbits of the vector field.

The study of singularities of implicit equations was one of the 4 problems in King Oscar II of Sweden's list of 1985. (One of the other problems was the 3-body problem and the prize-winning paper of Poincaré was his memoire on the 3-body problem).

The topological structure of the singular points of implicit ordinary differential equations was settled by A.V. Phakadze and A.H. Shestakov (1959); later this subject was studied independently by Thom (1972), L. Dara (1975), F. Takens (1976). These last three mathematicians have conjectured that (*) is a topological normal form (this fact was also proved by Phakadze and Shestakov and then rediscovered by Piliy and Fedorov, 1970, in the context of plasma physics).

The Davydov theorem implies that (*) is even a smooth and analytic normal form, which goes much further than all the preceding conjectures in this domain.

It is interesting to note that these mathematical results of contact geometry have many different interpretations in terms of different branches of mathematics and physics.

For instance, the same normal form describes the patterns of asymptotic lines at some points of a parabolic line on a generic surface in Euclidean 3-space.

It provides also the normal forms for the field of characteristics of a generic partial differential equation of mixed type, (whose type changes from elliptic to hyperbolic along some line in the plane of the independent variables).

One may also describe the same Davydov theorem in terms of the theory of relaxation oscillations in systems with two slow and one fast variables. In this case the contact structure is defined by a vertical (fast) vector field and by a small (slow) perturbing generic vector field in the 3-space of slow and fast variables. The Legendre fibration is defined by the (vertical) direction field of the fast motion. The surface in the contact space is formed by the equilibria points of the fast motion. It is usually called the slow surface in relaxation oscillation theory. The characteristics of this surface describe the orbits of the slow motion (occurring after the relaxation of the fast motion).

One more example of the application of the Davydov theorem is provided by the theory of Newton's equation with 1 degree of freedom $\ddot{x} = F(x) - k\dot{x}$.

Let us consider the (x, E) coordinate energy plane, $E = \dot{x}^2/2 + U(x)$, $F = -\partial U/\partial x$.

The projection of the motion to the (x, E)-plane defines a family of curves in the domain $E \geqslant U(x)$. At the points of the potential wells and of the barriers (of the minima and of the maxima of U) these curve families have singularities. The type of the singularity depends on the value of the friction coefficient k and on the type of the critical point of the energy. But the family is generically diffeomorphic to the family of integral curves in the (q, y) plane of Davydov's theorem.

Thus the general theorems of contact geometry unify many apparently different theories in many branches of mathematics and of physics, making transparent the common mathematical structures and features of all these theories. Hence the contact geometry strategy is to translate the intuition and concrete results of any of the branches of its application to all the other branches.

Let us consider a hypersurface in a contact manifold, which has no characteristic points. Such a hypersurface is foliated (locally fibrated) into characteristics. The set of characteristics is (at least locally) a manifold whose dimension is two less than the dimension of the initial contact manifold.

THEOREM. *The manifold of characteristics inherits a contact structure from the initial contact manifold.*

A formal way of proving this theorem is a direct calculation. According to the preceding formulae, the characteristics of the hypersurface $K = 0$ are the orbits of the vector field $W = V - K\partial/\partial z$, where $L_V\alpha = K_z\alpha$. Since $i_{\partial/\partial z}d\alpha = 0$, $i_{\partial/\partial z}\alpha = 1$, we have

$$L_W\alpha = K_z\alpha - dK .$$

The second term vanishes along the hypersurface $K = 0$. Thus the flow of the vector field W on the hypersurface $K = 0$ preserves the field of hyperplanes $\alpha = 0$ and hence defines a field of hyperplanes on the space of orbits of this field.

Another way of proving this theorem is to consider just one particular case, say the hypersurface, defined in Darboux coordinates by the equation $p_1 = 0$. In this case the characteristic direction is $\partial/\partial q_1$. Hence the space of the characteristics is the coordinate space with Darboux coordinates (z, p, q) where $\tilde{p} = (p_2, \ldots, q_n)$ $\tilde{q} = (q_2, \ldots, q_n)$. Thus the form $\alpha = dz + \dfrac{pdq + qdp}{2}$ induces on the manifold of characteristics of the hypersurface $p_1 = 0$ the form $\tilde{\alpha} = dz + \dfrac{\tilde{p}d\tilde{q} - \tilde{q}d\tilde{p}}{2}$, as was required.

Now the general case can be reduced to this particular case, since all the hypersurfaces in a contact manifold are locally contactomorphic in neighbourhoods of their non-characteristic points, which follows from the general theorem of Givental, described below.

§3. SUBMANIFOLDS

The submanifolds of a Euclidean or a Riemannian manifold have interior and exterior geometries. For instance, the Gaussian curvature belongs to the interior geometry of the Riemannian metric on the submanifold, while the mean curvature depends on its exterior geometry. In both symplectic and con-