§5. Legendre varieties and the obstacle problem

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§5. LEGENDRE VARIETIES AND THE OBSTACLE PROBLEM

According to a well known principle of A. Weinstein, everything in symplectic geometry "is" a Lagrangian manifold.

In contact geometry "everything" is a Legendre manifold. But the important examples suggest that in many applications smooth Lagrangian (Legendre) manifolds should be substituted by singular Lagrangian (Legendre) varieties.

In principle the translation of the general theory to the case of singular varieties (and even of the "schemes" of algebraic geometry) is routine work. But I believe that a more natural and useful notion of the Lagrangian (Legendre) singular varieties would be a generalization of the classification of the Givental "triads", presented below, rather than a theory of Legendre ideals of the hierarchy of degeneracies in the generating families.

Let us consider a medium, containing an obstacle (i.e. a manifold with a boundary). The fronts are the hypersurfaces, equidistant from a given one. For instance, if the obstacle is bounded by a plane curve, the fronts are its evolvents (involutes). Hence the following is a higher-dimensional generalization of the Huygens' involute theory.

A smooth front may acquire singularities while travelling through a smooth medium, but its Legendre manifold remains non-singular. At an obstacle, however, even the Legendre manifold may become singular. These singular *Legendre varieties* are singularly related to the irreducible finite-dimensional sl_2-modules. Namely, the singularities of the Legendre varieties at generic obstacles are diffeomorphic to those of the varieties of binary forms (or of polynomials in one variable) admitting roots of high multiplicity.

The simplicity of the final result is rather misleading: the polynomials and even their degrees are hidden. Even though they are known to exist it is still difficult to find them from geometrical considerations.

The relation of the obstacle problem to $sl(2)$ modules was discovered in 1981-82 as a result of a series of works featuring geometrical observations based on the resemblance of bifurcation diagrams, occurring in different theories, strange cancellations of many terms in long calculations, due to some properties of the varieties of polynomials with multiple roots, which seems to be new for the algebraists, and new concepts in symplectic and contact geometry, namely the "triads" of Givental, describing families of rays and of fronts at obstacle points.

Let us consider a smooth hypersurface ∂M in Euclidean space $M = \mathbf{R}^n$. Let us consider the length S of the shortest path from a fixed set to a variable end point, avoiding the obstacle, bounded by ∂M. The study of singularities of S as a function of the end point leads to the following problem.

Let us consider a family of geodesics on ∂M, orthogonal to some hypersurface of ∂M. The straight lines, tangent to these geodesics, define an $(n-1)$-parameter family of rays in $\mathbf{R}^n$, namely the family of all normals to some front hypersurface in $\mathbf{R}^n$. The problem is to study the singularities of these fronts.

Example 1. Let the obstacle be bounded by a generic plane curve ($n=2$). The fronts are the *involutes* of the curve. They have singularities of order $3/2$ at generic points of the curve (Huygens). A generic curve may have some inflection points. A calculation shows that the fronts have singularities of order $5/2$ at the points of the inflectional tangent. These singularities are related to the rather mysterious appearance of H_2 — of the symmetry group of a pentagon (at the point of the inflection it is replaced by the symmetry group H_3 of an icosahedron, but it is rather difficult to see this icosahedron in the neighbourhood of the inflection point with the naked eye).

Example 2. Let the obstacle be bounded by a generic surface in 3-space. The fronts are surfaces with cuspidal edges. These edges are of order $3/2$ at generic points of the boundary surface. Our one-parameter family of geodesics covers a domain in this surface. The geodesic direction may become asymptotic along some curve in this domain. The rays tangent to the geodesics at the points of this curve have asymptotic directions. The fronts' singularities at the points of an asymptotic ray are edges of order $5/2$, unless the ray is "bi-aymptotic" (this may happen at some points of our curve).

The theory described below explains the contact geometry of these complicated singularities and of their higher dimensional counterpart in terms of the theory of invariants of sl_2.

Definition. A *contact triad* (H, L, l) consists of

(i) a noncharacteristic hypersurface H in a contact manifold,

(ii) a Legendre submanifold L in the same contact manifold,

(iii) a smooth hypersurface l in L

such that the hypersurface H is tangent to the Legendre manifold L with first order of tangency at every point of l.

We shall study the germ of a triad at a point 0 of l.

Definition. The *Legendre variety, generated by the triad* at 0 is the image of the germ of l at 0 by the projection of H onto its space of characteristics.

Example. Let us consider the family of geodesics on a hypersurface $\partial M \subset M = \mathbf{R}^n$, consisting of all geodesics, normal to a surface of codimension 1 in ∂M. We shall associate to this family a contact triad.

Let H be the hypersurface of light contact elements of the space-time $\mathbf{R}^n \times \mathbf{R} = \{q, t\}$, defined by $dt = pdq$, $p^2 = 1$. This hypersurface is the first element of the triad.

Let $s \colon \partial M \to \mathbf{R}$ be the time function, defining the family of geodesics on ∂M (the geodesics are orthogonal to the surfaces $s =$ const, and $(\nabla s)^2 = 1$ on ∂M). The graph of s is a codimension 2 submanifold of space-time. Let us consider the set of all space-time contact elements, tangent to this graph. This set is a Legendre manifold and it will serve as the L of the triad.

THEOREM. *The hypersurface H is (first order) tangent to the Legendre manifold L along a submanifold l of codimension 1 in L, consisting of all contact elements, belonging to L, which contain the normal to $\partial S \times t$ in $\mathbf{R}^n \times t$.*

The Legendre variety, generated by this triad, is formed by those contact elements of $\mathbf{R}^n$, which are tangent to the same front for the obstacle problem with boundary ∂M (and initial condition s).

The theorem follows almost immediately from the analysis of the hexagonal diagram in §3. For more details see [A5] and [A6].

We will now construct a series of examples of triads, providing normal forms for the germs of generic triads at all their points. This implies, for instance, normal forms of singularities of the Legendre varieties, consisting of all contact elements tangent to a front for a generic obstacle in a Euclidean or Riemannian space.

We start with the natural sl_2-invariant contact structure of the projective space of the 0-dimensional hypersurfaces of degree $d = 2n - 1$ on the projective line (§1, example 7).

Let us consider maps of the space of polynomials of the form

$$X_{2n-1} + q_2 X_{2n-2} + \dots + q_n X_n - p_n X_{n-1} + \dots \pm p_1 ,$$

where $X_j = x^j/j!$, with the contact structure $\alpha = 0$, where

$$\alpha = p'dq' - q'dp' - dp_1 , \quad p' = (p_2, \dots, p_n) , \quad q' = (q_2, \dots, q_n) .$$

The group of translations of polynomials along the x-axis acts on the space of these polynomials and preserves its contact structure. Let v be the corresponding vector field. We define the *neutral surface* H by the equation $\alpha \mid v = 0$. An explicit calculation gives the equation of the neutral hypersurface

$$H: K \equiv p_2 + p_3 q_2 + \dots + p_n q_{n-1} + q_n^2/2 = 0$$

(known essentially to Hilbert).

This formula implies the obvious

LEMMA. *The triple* $H(K=0)$, $L(p=0)$, $l(p=q_n=0)$ *is a contact triad.*

The Legendre variety Σ, generated by this triad, consists of those polynomials

$$X_{2n-1} + q_3 X_{2n-3} + \dots + q_n X_n - p_n X_{n-1} + \dots + (-1)^n p_1 ,$$

where $X_j = x^j/j!$ and $p_2 = -(p_4 q_3 + \dots + p_n q_{n-1} + q_n^2/2)$, which have a root of multiplicity greater than n; the contact structure is defined by the 1-form

$$\alpha = p'' dq'' - q'' dp'' - dp_1 , \quad p'' = (p_3, \dots, p_n) \quad q'' = (q_3, \dots, q_n) .$$

The Legendre variety Σ^m of dimension $m = n - 2$ thus defined will be called the *Givental Legendre variety.* It lives in a contact space of dimension $2m + 1$.

These varieties (and their Lagrangian projections to symplectic spaces, also studied by Givental) have remarkable properties, both as algebraic varieties and as contact (or symplectic) space subvarieties.

Let us first describe them as algebraic varieties. We start with the tower of spaces of polynomials in one variable x, equipped with the projection given by the derivative $D = (n+1)^{-1}(d/dx): \mathbf{C}^n \to \mathbf{C}^{n-1}$,

$$\mathbf{C}^n = \{x^{n+1} + A_1 x^{n-1} + \dots A_n\} ,$$

(one may consider as well the tower of the spaces of polynomials $x^{n+1} + A_0 x^n + \dots$ or even $Ax^{n+1} + A_0 x^n + \dots$).

Let us consider a root of multiplicity m of a polynomial of degree d.

The number $d - m$ is called the *comultiplicity* of a root. The spaces of polynomials are "stratified" according to the comultiplicities. We denote the set of polynomials $x^{n+1} + A_1 x^{n-1} + \dots + A_n$ having a root of comultiplicity at most m by $\Sigma_m(n) \subset \mathbf{C}^n$. $\Sigma_m(n)$ is an algebraic variety of dimension m.

Example. $\Sigma_1(2)$ is the discriminant curve in the plane of cubical polynomials $x^3 + A_1 x + A_2$, $\Sigma_1(3)$ is the cusped edge of the swallowtail, $\{x^4 + B_1 x^2 + B_2 x + B_3 = (x+t)^3 \dots\}$.

THEOREM. *The derivation mapping* $D: \mathbf{C}^n \to \mathbf{C}^{n-1}$ *preserves the comultiplicities, provided that the multiplicity is greater than one:* $D\Sigma_m(n) = \Sigma_m(n-1)$ *for* $m < n$.

Moreover, this mapping is a diffeomorphism, provided that Σ_m *has no selfintersections, namely for* $n > 2m$.

Example 1. The cusped edge of the swallowtail, $\Sigma_1(3)$, projects diffeomorphically on the plane semicubical parabola $\Sigma_1(2)$.

Example 2. The surface $\Sigma_2(4) \subset \mathbf{C}^4$ projects onto the usual swallowtail surface $\Sigma_2(3)$, but this mapping is not one-to-one, since generic points of the selfintersection line of the swallowtail surface have two counterimages.

If we start with the generalized swallowtail $\Sigma_m(m+1)$ of dimension m in $\mathbf{C}^{m+1}$ and go up the storeys of the tower, we obtain a sequence of m-dimensional varieties and of projections

$$\dots \Sigma_m(2m+1) \to \Sigma_m(2m) \to \dots \to \Sigma_m(m+1)\,.$$

The sequence stabilizes at the floor of $\Sigma_m(2m) \subset \mathbf{C}^{2m}$, where the last selfintersection disappears and the variety becomes homeomorphic to $\mathbf{C}^m$ (A.B. Givental).

Example. The ordinary swallowtail $\Sigma_2(3)$ is stabilized at the next floor, where lives the *open* (or "*unfurled*") *swallowtail* $\Sigma_2(4)$. This surface is stable ($\Sigma_2(4) \approx \Sigma_2(5) \approx \dots \approx \Sigma_2(\infty)$).

The importance of the open swallowtail for variational problems was discovered in 1981 (see [A3] and [A4]).

Remark. The diffeomorphism $\Sigma_m(2m+1) \to \Sigma_m(2m)$ is induced by a section of the mapping $D: \mathbf{C}^{2m+1} \to \mathbf{C}^{2m}$, which is a paraboloid. The equation of this paraboloid was found by Hilbert (1893). It follows from his

LEMMA. *Let*

$$x^n + \lambda_1 x^{n-1} + \dots + \lambda_n = (x-a)^{k+2}(x^k + a_1 x^{k-1} + \dots + a_k)\,.$$

Then

$$2n!\lambda_n = \Sigma(-1)^{i+1} i!(n-i)!\lambda_i\lambda_{n-i} \qquad 1 \leqslant i < n\,.$$

The next floors also admit parabolic sections. Let us define the operators $G^{(s)}$ by the formula

$$G^{(s)}[F] = (d^sF/dx^s)/(s-r)!$$

where r is fixed and where the factorials of negative numbers are defined by a truncation at some sufficiently remote place, say

$$(s-r)! = (s-r) \dots (1-r) .$$

THEOREM. *If* $F = (x-a)^{r+1}(x^{m-1}+\dots)$, *where* $r > m$, *then*

$$\Sigma(j-i)G^{(i)}[F]G^{(j)}[F] = 0$$

(summation over $i+j=r+m, 0 \leqslant i \leqslant m$*).*

This theorem implies the stabilization property.

For further details on the stabilization and on the general theory of unfurled swallowtails see [G1] and [A6].

For instance, the modules of vector fields, tangent to the unfurled swallowtails have been studied by Givental, who has proved the

THEOREM. *Any germ of a holomorphic vector field, tangent to the generalized swallowtail* $\Sigma_m(m+1)$ *at the origin, is the projection of a germ of a vector field in* $\mathbf{C}^{2m}$ *tangent to its stabilization* $\Sigma_m(2m)$.

Any polynomial vector field, tangent to $\Sigma_k(n)$, may be represented as a sum of a vector field, tangent to the swallowtail $\Sigma_{n-1}(n)$ and of vector fields whose projections to $\mathbf{C}^{n-1}, \dots, \mathbf{C}^{k+1}$ are vector fields tangent to the projections $\Sigma_k(n-1), \dots, \Sigma_k(k+1)$ of the original variety.

The Givental theory implies that the unfurled swallowtail $\Sigma_m(2m)$ is a Lagrangian subvariety of the space of polynomials $x^{2m+1} + a_1x^{2m-1} + \dots + a_{2m}$ equipped with its natural symplectic structure

$$\Sigma(-1)^i i!j!\, da_i \wedge da_j , \quad i + j = 2m - 1 .$$

(This structure is natural in the sense that it is naturally derived from the sl_2-invariant symplectic form on the space of binary forms of odd degree.)

Example. The ordinary (two-dimensional) unfurled swallowtail $\Sigma_2(4) = \{x^5 + Ax^3 + Bx^2 + Cx + D = (x+t)^3 \dots\}$ is a Lagrangian subvariety for the natural symplectic structure

$$3dA \wedge dD - dB \wedge dC .$$

The unfurled swallowtails describe the singularities and the perestroikas of the duals of the projective space curves. Let $\phi: \mathbf{R} \to \mathbf{R}^3$ be a smooth map. We call (a, b, c) its *type*, if the first of its derivatives which is nonzero at the origin, is the a-th one, the first noncolinear with it — the b-th, the first noncoplanar with them — the c-th.

The curves of types $(a_1, \ldots, a_n)$ in $\mathbf{R}^n$, $\mathbf{RP}^n$, $\mathbf{C}^n$ or $\mathbf{CP}^n$ are defined by a similar construction of an osculating flag. Any curve of a finite type has an osculating hyperplane at every point.

Definition. The *dual curve* $\phi^\vee : \mathbf{R} \to \mathbf{RP}^{n\vee}$ of a curve $\phi : \mathbf{R} \to \mathbf{RP}^n$ is the set of osculating hyperplanes of ϕ. Of course, $\phi^{\vee\vee} = \phi$ and the dual type of $(a_1, \ldots, a_n)$ is $(a_n - a_{n-1}, \ldots, a_n - a_1, a_n)$.

The hierarchy of the smooth $(a_1 = 1)$ curves in $\mathbf{R}^3$ starts with

$$\begin{array}{ccccccc}
\underset{\text{(ordinary point)}}{(1, 2, 3)} & \leftarrow & \underset{\text{(flattening)}}{(1, 2, 4)} & \leftarrow & (1, 2, 5)\ \text{(biflattening)} & \leftarrow & \ldots \\
 & & & \nwarrow & & & \\
 & & & & (1, 3, 4)\ \text{(inflection)} & \leftarrow & \ldots
\end{array}$$

The codimension of the type is $c = \Sigma(a_i - i)$.

The set of all the hyperplanes, containing the tangent line of a projective curve ϕ, forms a developing hypersurface in the dual space, having a cusped edge $\phi^\vee$. We call this developing hypersurface *the front of* ϕ. (This is a particular case of the general definition of the front of a submanifold of projective space as of fronts of the corresponding Legendre mapping. We start with any submanifold M in $\mathbf{RP}^n$, and construct the Legendre submanifold L formed by the contact elements of $\mathbf{RP}^n$, tangent to M, and we project L to the dual space $\mathbf{RP}^{n\vee}$ along the fibres of the Legendre fibration:

$$M^k \qquad L^{n-1} \to PT^*\mathbf{RP}^n = PT\mathbf{RP}^{n\vee} \twoheadrightarrow \mathbf{RP}^{n\vee}.$$

The resulting Legendre mapping $L^{n-1} \to \mathbf{RP}^{n\vee}$ is called the *frontal mapping of M*).

Example. Let the curve ϕ have a simplest flattening $(1, 2, 4)$. Then the curve $\phi^\vee$ has a singularity of type $(2, 3, 4)$ like (t^2, t^3, t^4). The tangent lines sweep out the usual swallowtail which is the front of ϕ.

A generic curve in $\mathbf{R}^3$ has isolated flattening points but has no more complicated degeneracies. Hence a front of a generic space curve is a surface with a cusped edge and with isolated swallowtails.

In a 1-parameter family of curves in $\mathbf{R}^3$ the bi-flattening points become unavoidable (for some exceptional values of the parameter). The family of dual curves in a family of dual spaces forms a surface in a 4-space.

THEOREM (0. Shcherback). *This surface is locally diffeomorphic to the ordinary unfurled swallowtail and its decomposition into dual curves is dif-*

feomorphic to the decomposition of the space of polynomials $x^5 + Ax^3 + Bx^2 + Cx + D$ *having a triple root into the "isochrones", defined by the equation* $A = const.$

The inflection points give rise to the surface $\{x^4 + Ax^3 + Bx^2 + Cx + D$ has a multiple root$\}$ and to the "isochrones" $B =$ const.

The general theory of Shcherbak is described in his paper [Sh1].

For instance, he has proved the following theorems.

1. A front of a curve, dual to a generic smooth space curve (that is, the union of the tangents of the smooth curve) has at the (1, 2, 4) inflection point of the original smooth curve a "folded Whitney umbrella", locally diffeomorphic to the germ of the surface $x^2y^3 = z^2$ at the origin.

2. The singularities of the front of a generic smooth curve in $\mathbf{RP}^n$ are locally diffeomorphic to the discriminants A_k, i.e. to the products of generalized swallowtails and of smooth manifolds. The union of the front of a curve with the hyperplane, dual to a point of the initial curve, is locally diffeomorphic to a discriminant of Lie algebra B_k.

3. In typical one-parameter families of curves in $\mathbf{RP}^n$ there exist unavoidably isolated points of types $(1, \ldots, n-1, n+2)$ and $(1, \ldots, n-2, n, n+1)$. The corresponding fronts' perestroika patterns are

$$\{x^{n+2} + \lambda_1 x^n + \ldots + \lambda_{n+1} = (x+t)^2 \ldots\}, \quad \lambda_1 = \text{const.};$$
$$\{x^{n+1} + \lambda_1 x^n + \ldots + \lambda_{n+1} = (x+t)^2 \ldots\}, \quad \lambda_2 = \text{const.}$$

4. Any stable Legendre singularity of corank m is Legendre equivalent to a germ of a frontal mapping of an m-dimensional submanifold of a projective space. The frontal mappings of the generic space curves have only A_k singularities. The perestroikas of the Legendre mappings of corank m in the generic families with finite dimensional parameter spaces are Legendre equivalent to the perestroikas of the frontal mappings of the m-dimensional submanifolds of projective spaces.

The latest development of the contact geometry of projective space curves is due to M. Kazarjan (1985; see his paper [Ka]). In his theory the higher codimension degenerations are classified by their Young diagrams, and related to the singularities of the Schubert cell decomposition of a Grassmannian manifold.[1])

[1]) The Schubert cells of flag manifolds are related to Tchebychev systems and to nonoscillating linear ODE, as was discovered by B.Z. Shapiro (1985).

The Kazarjan theory introduces some new remarkable bifurcation diagrams. For instance, the Young diagram (2, 2) corresponds to a union of two Whitney umbrellas, tangent cubically along two lines. One of the ways of visualizing this bifurcation diagram is due to Shcherbak. Let us consider a generic 1-parameter family of projections of a smooth curve to a plane. For some isolated values of the parameter the projection has a cusp. The corresponding perestroika is of type "from γ to u". The projections may also be described as the generic sections of a Whitney umbrella by " isochrones" (parallel planes).

Now let us add to each of these plane curves their inflectional tangents. These tangents sweep out a surface in 3-space. This surface is the second Whitney umbrella. The two umbrellas together from the Kazarjan bifurcation diagram. The general theory of such diagrams and of their applications to different problems of calculus, optics and so on may be found in Kazarjan's paper quoted above and in the 2 volumes of the Springer Mathematical Encyclopaedia, devoted to singularities (the first of these is the volume "Dynamical systems — 6", Moscow VINITI 1988, but we need the second, VINITI 1989, "Dynamical systems – 8", Encyclopaedia vol. 39).

Let us now return to the obstacle problem. Comparing the explicit formula defining the Givental Legendre variety and the definition of the unfurled swallowtails, we obtain

THEOREM. *The Givental Legendre variety of dimension* m *is diffeomorphic to the unfurled swallowtail* $\Sigma_m(2m) = \Sigma_m(\infty)$.

Example. The 2-dimensional Givental Legendre variety lives in the contact 4-space of polynomials

$$x^7/7! + q_3x^5/5! + q_4x^4/4! + p_4x^3/3! - p_3x^2/2! + ux - p_1 ,$$

where

$$u = -(q_3p_4 + q_4^2/2), \qquad \alpha = p_3dq_3 + p_4dq_4 - q_3dp_3 - q_4dp_4 - dp_1 ,$$

and consists of polynomials, which have a root of multiplicity 5 or greater. It is diffeomorphic to $\Sigma_2(4)$.

THEOREM. *The standard contact triads defining the Givental Legendre varieties, are stable (as germs of contact triads considered up to contact equivalence). The germs of generic contact triads are contactoequivalent to the germs of the standard ones.*

COROLLARY. *The Legendre varieties defined by generic contact triads are locally contactomorphic to the Givental Lagrange varieties.*

COROLLARY. *The germs of Legendre varieties, formed by contact elements, tangent to a front in the generic obstacle problem are contact stable and contact equivalent to the germs of the Givental Legendre varieties (and hence are diffeomorphic to the germs of the unfurled swallowtails).*

Returning to the hexagonal diagrams of §3, we can now find the normal forms of the Legendre varieties describing the "multivalued time functions".

THEOREM. *The 1-graph (the set of 1-jets) of the time function in a generic obstacle problem in* $\mathbf{R}^m$ *is locally contactomorphic to the Legendre variety formed by the polynomials* $(x-\xi)^{m+2}(x^{m-1}+\ldots)$ *in the space of polynomials of degree* $n = 2m-1$

$$x^n + a_1 x^{n-1} + \ldots + a_n ,$$

equipped with its natural (sl_2-invariant) contact structure $\Sigma i!j!(-1)^i a_i\, da_j = 0$ *(where* $i+j=n,$ $a_0=1$*).*

The time function restriction to this variety is $\pm a_1 + \text{const};$ *the points of the same ray correspond to the translations of a polynomial along the* x *axes.*

COROLLARY. *The variety of the contact elements of a generic front in the obstacle problem in* $\mathbf{R}^m$ *is locally contactomorphic to the Legendre variety.*

$$\{\lambda : \exists \xi : x^n + \lambda_{n-1}x^{n-1} + \ldots + \lambda_0 = (x-\xi)^m(x^{m-1}+m\xi x^{m-2}+\ldots)\ \forall x\}$$

in the space of polynomials of degree $n = 2m-1$ *equipped with the perverse contact structure*

$$d\lambda_{n-1} = \Sigma i!j!(-1)^i(i+2)\lambda_i d\lambda_j, \quad 0 \leqslant i \leqslant n-2, \quad i+j = n-2 .$$

Now to derive the singularities of the fronts and of their perestroikas in the obstacle problem we have to project the above normal forms to the base of a Legendre fibration (for instance, from the space J^1 of 1-jets of functions to the space J^0 of 0-jets, where the ordinary graph of the time function lives). This is unfortunately a difficult problem (it is discussed in Givental's thesis, and in his paper [G2]).

The classification of the generic front singularities in the obstacle problem was obtained by O.P. Shcherbak in another way in 1984 (see, for instance, Uspekhi Math. Nauk 1984, vol. *39,* N 5, p. 256[1])). Unfortunately, the final text of his proofs appeared only after he died in 1985.

I am very grateful to I.G. Shcherbak and to A.B. Givental who have prepared O.P. Shcherbak's manuscript "Wave fronts and reflection groups" for publication. It has finally appeared in [Sh3].

The main discoveries of Shcherbak in this paper are the local diffeomorphisms of the fronts and the graphs of the multivalued time function in the generic obstacle problem to the discriminants of the noncrystallographic Coxeter groups H_2, H_3, H_4.

H_2 is the symmetry group of a pentagon. Its orbit space is $\mathbf{C}^2$, and the irregular orbits form a discriminant curve with a singularity $x^2 = y^5$. It was perhaps known to Huygens and it is written explicitly in the book of L'Hospital (1696), that this singularity appears at the inflection tangent to a generic plane curve as a singularity of the involute (that is, of the generic front in a two-dimensional obstacle problem).

H_3 is the symmetry group of an icosahedron. The orbit space is $\mathbf{C}^3$ and the discriminant surface has been studied by O.V. Lyashko with the help of a computer. A.B. Givental (1982) recognized in this picture the graph of the multivalued time function of the plane obstacle problem, which I had shown him a year before. Then O.P. Shcherbak proved the Givental conjecture: the germ of the multivalued time function at the generic inflection point of the obstacle is locally diffeomorphic to the surface of irregular orbits of H_3. One may find the proofs in the papers [Ly] and [Sh2].

H_4 is the symmetry group of a convex polyhedron with 120 vertices in $\mathbf{R}^4$. To describe this polyhedron we start from the rotation group of an icosahedron which contains 60 elements. The double covering $S^3 \to SO(3)$ lifts this group to a subgroup of 120 elements in S^3. Those 120 points of S^3 form the vertices of our polyhedron.

In his study of the singularities of the fronts and time function graphs in the obstacle problem, O.P. Shcherbak has found among other things the discriminant of H_4. Namely it is the singularity of the graph of the multivalued time function at some "focal" point of a tangent line to a geodesic of the family of geodesics on the surface of the obstacle (defined by the initial condition). The tangent line itself is very special: it is an asymptotic line of the surface in one of its parabolic points (for a generic family of geodesics the direction of the geodesics changes along the parabolic line of the surface and

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[1]) "Uspekhi" are translated by the London Mathematical Society as "Russian Math. Surveys". But some pages of the Uspekhi contain news and announcements and hence are not translated.

consequently becomes asymptotic at some isolated points of the parabolic line; these points depend on the family).

The proof depends on a classification of the families of functions with critical points of only even multiplicity. The simple germs of this type are

$$A'_{2k}: y^2 + \int_0^x (u^k + q_1u^{k-2} + \ldots + q_{k-1})^2 du + q_k;$$

$$D'_{2k}: \int_0^y (u^{k-1} + q_1u^{k-3} + \ldots + q_{k-3}u + x)^2 du + q_{k-2}x^2 + q_{k-1}x + q_k;$$

$$E'_6: x^3 + y^4 + q_1y + q_2y + q_3;$$

$$E'_8: x^3 + y^5 + q_1y^3 + q_2y^2 + q_3y + q_4;$$

$$E''_8: x^3 + \int_0^y (u^2 + q_1x + q_2)^2 du + q_3x + q_4.$$

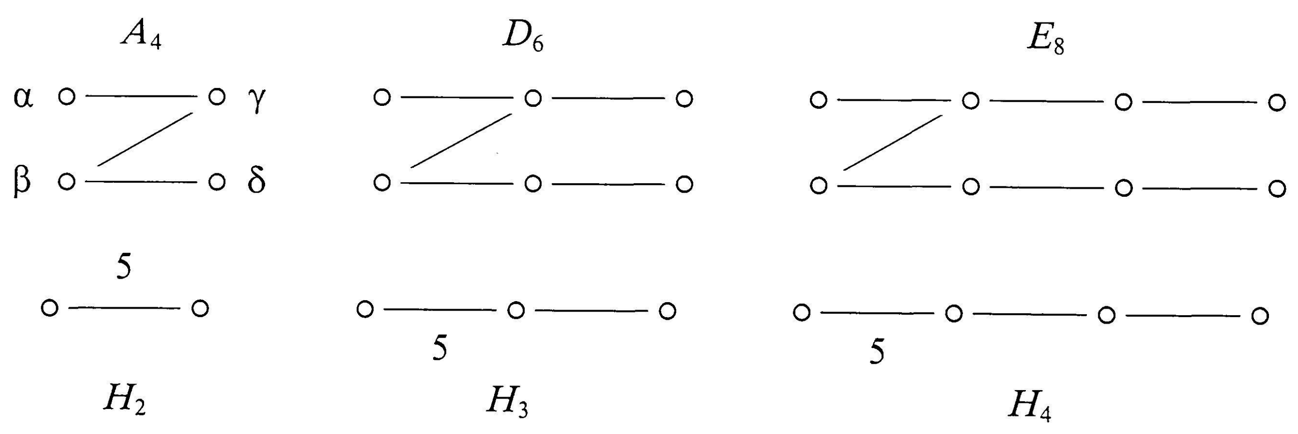
The front of the family is the set of parameters q, such that 0 is a critical value.

THEOREM. *The optical length in the generic obstacle problem in* $\mathbf{R}^3$ *(considered as a family of functions of the initial point depending on the final point as on a parameter) has only the simple critical points of the preceding list. Hence the graph of the multivalued time function ("the big front") is locally diffeomorphic to the Cartesian product of the front of one of the families* $A'_2, A'_4, A'_6, D'_6, E'_6, D'_8, E'_8, E''_8$ *with a non-singular manifold.*

From A'_4 one obtains the 5/2 singularity H_2, from D'_6 the icosahedral discriminant H_3. This is the singularity of the front at the generic points of the surface, where the geodesic has an asymptotic direction.

The discriminant of the group H_4 is diffeomorphic the front of the family E''_8

The paper of Shcherbak contains a lot of information on these and other Coxeter groups. It is interesting to note that the "foldings" $A_4 \to H_2$, $D_6 \to H_3$, $E_8 \to H_4$ may be defined by the unusual forms of the Dynkin diagrams



Let us first describe the folding $A_4 \to H_2$. The reflections, corresponding to α and to β commute and their product is an element of the A_4 reflection group. In the same way $\gamma\delta$ defines another element, and these two elements generate a subgroup in A_4. This subgroup is a representation of the pentagon symmetry group H_2 in A_4. It is reducible and $\mathbf{R}^4$ is decomposed into a direct sum of two 2-planes invariant under H_2. This construction of an irrational subspace in the space $\mathbf{R}^4$ with a lattice A_4 everything being invariant under the 5-fold symmetry of H_2, allows us to define in $\mathbf{R}^2$ the quasiperiodic Penrose tilings having H_2 symmetry (for the details see [A7]).

The same way the folding $D_6 \to H_3$ generates a subspace $\mathbf{R}^3 \subset \mathbf{R}^6$, invariant under the action preserving the D_6-lattice of the icosahedral symmetry group H_3. This way we construct quasicrystals in $\mathbf{R}^3$ with the icosahedral symmetries.

Finally, the construction of H_4 from E_8 defines in $\mathbf{R}^4$ quasicrystals with the 120×120 symmetries of H_4.

Since the spaces $\mathbf{R}^4$, $\mathbf{R}^6$, $\mathbf{R}^8$ and their lattices A_4, D_6, E_8 may be interpreted as the homology of the corresponding Milnor fibres with $\mathbf{R}$ or $\mathbf{Z}$ coefficients, we obtain some special functions, associated to H_2, H_3, H_4 (generalizing the Airy function associated to A_2, the Piercy function, associated to A_3 and so on, see [VC]).

There exists one more series of noncrystallographical Coxeter groups, $I_2(p)$.

A.B. Givental has discovered a problem in contact geometry, whose solutions are in a one-to-one correspondence with the Coxeter Euclidean reflection groups. This is the problem of the Legendre classification of the simple stable Legendre singularities, whose Legendre varieties are diffeomorphic to the products of curves (or of at most one singular curve with a smooth manifold). His treatment of the series $I_2(p)$ is based on the multiple folding of an A-diagram (see [G2]).