THE CATEGORY OF NILMANIFOLDS

Autor(en): **Oprea, John**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **38 (1992)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **15.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-59480

L'Enseignement Mathématique, t. 38 (1992), p. 27-40

THE CATEGORY OF NILMANIFOLDS

by John OPREA

ABSTRACT. The techniques of rational homotopy theory are used to compute the category of a nilmanifold: $\operatorname{cat}(M) = \dim M = \operatorname{rank}(\pi_1 M)$. This information is of interest to dynamicists since the theorem of Lusternik-Schnirelmann then shows that the number of critical points of a smooth function of M is bounded below by $\operatorname{rank}(\pi_1 M) + 1$.

INTRODUCTION

As a first step to understanding the structure of certain dynamical systems on nilmanifolds, one might hope to have computable lower bounds on the number of critical points of smooth functions. Of course, one is then led to the Lusternik-Schnirelmann definition of category and their well-known result that category $(+1)$ is such a bound. Unfortunately, category is rarely computable, so those who require numerical bounds often employ the fact that category majorizes cuplength. Hence cuplength (which, generally, is a more computable homotopy invariant than category) is the numerical invariant frequently sought for in order to provide a lower bound for the number of critical points of smooth functions on a manifold.

Indeed, some time ago, for the reasons above, Chris McCord asked me if I knew of a formula for the cuplength of a nilmanifold. I did not then, and after many computations I do not now! Thus, I pose:

QUESTION. What is the cuplength (with $\mathbf{Q}$-coefficients say) of a nilmanifold?

Suprisingly, however, the need for such knowledge by dynamicists is obviated by the following.

THEOREM 1. *If* M *is a (compact) nilmanifold, then* $\mathrm{cat}(M) = \dim(M) = \mathrm{rank}(\pi_1 M)$.

Hence, the best possible result which Lusternik-Schnirelmann theory can provide for nilmanifolds is the immediate.

COROLLARY. *The number of critical points of a smooth function on a (compact) nilmanifold* M *is bounded below by* $\mathrm{rank}(\pi_1 M) + 1$.

In fact, Theorem 1 was announced for all $K(\pi, 1)$'s by Eilenberg and Ganea [11]. Unfortunately, details of the proofs of their three fundamental propositions never appeared, thus contributing, I believe, to the ignorance of the result among the dynamicists and topologists of today. Indeed, this paper was originally written in response to Chris McCord's question and without knowledge of the Eilenberg-Ganea result. Furthermore, in looking at the Eilenberg-Ganea propositions, it is difficult to see the relationship between the structures of π and $K(\pi, 1)$ and the consequent determination of category as $\mathrm{rank}(\pi)$. I hope that the approach of this paper will remedy this defect, at least in the case of nilmanifolds. The beautiful structure theory of nilmanifolds (i.e. finitely generated torsionfree nilpotent groups) is ideally suited for an approach in terms of minimal models. In fact, in some sense, this paper is simply an exposition of just how well rational homotopy theory and nilmanifold theory fit together (in the representative situation of determining category).

Theorem 1 will be given a simple ("up to" the machinery of rational homotopy theory) proof in §4. Since this paper is written for workers in dynamical systems, I have tried to make it somewhat self-contained. Therefore, §1 and §2 are devoted to recollections on category and its rational homotopy description respectively. §3 recollects structural knowledge of nilmanifolds and §5 presents an analogue of Theorem 1 for iterated principal bundles. (The basic reference for the rational homotopy version of L.S. category is [3]; I have attempted to cull the essential ingredients for the proof of Theorem 1, but the reader will find other interesting applications in that work. Also see [2].)

§1. CATEGORY

The *category* of a space M, $\mathrm{cat}(M)$, is the least integer m so that M is covered by $m + 1$ open subsets each of which is contractible within M.

An equivalent definition (at least for the spaces we consider here) was given by G. Whitehead (see [10]): Let M^{m+1} denote the $(m + 1)$-fold product and let $T^{m+1}(M)$ denote the subspace consisting of all $(m+1)$-tuples $(x_1, \cdots, x_{m+1})$ with at least one x_i equal to a specified basepoint in M. ($T^{m+1}(M)$ is usually called the "fat wedge".) In particular, $T^2(M) = M \vee M$; two copies of M attached at the specified basepoint. Now let $\triangle : M \to M^{m+1}$ denote the $(m+1)$-fold diagonal $\triangle(x) = (x, x, \cdots, x)$ and $j: T^{m+1}(M) \to M^{m+1}$ the natural inclusion. Whitehead's definition is then: $\mathrm{cat}(M)$ is the least integer m so that, up to homotopy, $\triangle$ factors through the fat wedge; that is, there exists $\triangle' : M \to T^{m+1}(M)$ with $j\triangle' \simeq \triangle$.

The *cuplength* of M, $\mathrm{cup}(M)$, is the largest integer k so that there exist $x_i \in H^{n_i}(M; R)$, $i = 1, \cdots, k$ and a nontrivial cup-product

$$0 \neq x_1 x_2 \cdots x_k .$$

The following result is well-known and is the basis of many calculations of category:

PROPOSITION. $\mathrm{cup}(M) \leq \mathrm{cat}(M)$.

For a proof, see [10] for example. Other important properties of category are:

(1) Category is an invariant of homotopy type.

(2) If $C_f = Y \cup_f CX$ is a mapping cone, then $\mathrm{cat}(C_f) \leq \mathrm{cat}(Y) + 1$.

(3) If X is a CW-complex, then (by induction on skeleta and (2)) $\mathrm{cat}(X) \leq \dim X$.

(4) In fact, (3) may be generalized: If X is $(r-1)$-connected, then $\mathrm{cat}(X) \leqslant (\dim X)/r$.

The proofs of these properties are straightforward; see [10] for example. In particular, we shall use (3) in our determination of the category of nilmanifolds.

Examples

1. $\mathrm{cat}(X) = 0$ if and only if X is contractible.

2. $\mathrm{cat}(S^n) = 1$.

3. More generally, $\mathrm{cat}(X) = 1$ if and only if X is a nontrivial co-H space.

4. $\mathrm{cat}(T^n) = n$ (this follows from the proposition and property (3) above).

We single out an example of interest in dynamical systems which, although quite simple, does not seem to be well known among dynamicists. (The analogue for Kähler manifolds *is* well known among topologists.)

5. If M^{2n} is a simply connected compact symplectic manifold, then $\mathrm{cat}(M) = n = \frac{1}{2}\dim(M)$. (First, observe that the volume form is not exact since it represents a nontrivial fundamental class of M. Because $\omega^n/n! = \mathrm{vol}$ (see [1], p. 165), the nondegenerate closed 2-form ω cannot be exact either. Hence, ω^n represents a nontrivial cup-product of length n in $\mathbf{R}$-cohomology. By property (4) above, $\mathrm{cat}(M) \leqslant (\dim M)/2 = n$. Hence,

$$n \leq \mathrm{cup}(M) \leq \mathrm{cat}(M) \leq \frac{1}{2}\dim M = n ,$$

and the result follows.)

§2. Rational homotopy and category

The basic reference for this section is [3]. To each space X, Sullivan functorially associated a *c*ommutative *d*ifferential *g*raded *a*lgebra $(A(X), d)$ of rational polynomial forms possessing the salient property that integration defines a natural algebra isomorphism between $H^*(A(X), d)$ and $H^*(X; \mathbf{Q})$. Furthermore, the cdga $A(X)$ was shown to contain all the rational homotopy information about X; information which may be gleaned from an associated cdga *minimal model* of $A(X)$.

A cdga (Λ, d) is *minimal* if (1) $\Lambda = \Lambda X$, where $X = \oplus_{i>0} X^i$ is a graded $\mathbf{Q}$-vector space and ΛX denotes that Λ is freely generated by X; that is, $\Lambda X = $ Symmetric algebra $(X^{\mathrm{even}}) \otimes$ Exterior algebra (X^{odd}). (2) There is a basis for X, $\{x_\alpha\}_{\alpha \in I}$, so that if I is well ordered by $<$, then $dx_\beta \in \Lambda^+_{\alpha<\beta}(x_\alpha) \cdot \Lambda^+_{\alpha<\beta}(x_\alpha)$. That is, Λ is constructed by stages and the differentials of β^{th} stage generators are decomposable in the generators of previous stages.

A *minimal model* for a space M is a minimal cdga $\Lambda(M)$ and a cdga map $\Lambda(M) \to A(M)$ inducing an isomorphism in cohomology. The fundamental theorem of rational homotopy theory is then (see [4] for example).

THEOREM. *Each space M has a minimal model $\Lambda(M)$ and, furthermore, for nilpotent spaces the stage by stage construction precisely mirrors the rational Postnikov tower with the differential corresponding to the k-invariant.*

Recall that a space M is *nilpotent* if its fundamental group $\pi_1(M)$ is a nilpotent group and the natural action of $\pi_1(M)$ on $\pi_n(M)$ (see [10]) is a nilpotent action (see [12]). In particular, any simply connected space or any $K(\pi, 1)$ with π nilpotent is a nilpotent space. The theorem then says that, for a nilpotent space, the minimal model is a perfect reflection of the rational homotopy type of the space (eg for $i > 1$, $X^i \cong \mathrm{Hom}(\pi_i(M), \mathbf{Q})$, where $\pi_i(M)$ is the i^{th} homotopy group of M). The minimal model $\Lambda(M)$ is therefore an algebraic version of the $\mathbf{Q}$-localization of M. Indeed, a notion of cdga homotopy may be described so that there is a categorical equivalence between the homotopy categories of rational nilpotent spaces and minimal cdga's.

Examples. (1) $\Lambda(S^{2n+1}) = \Lambda(x_{2n+1}), dx = 0$.

(2) $\Lambda(S^{2n}) = \Lambda(x_{2n}, y_{4n-1}), dy = x^2$.

(3) $\Lambda(\mathbf{C}P(n)) = \Lambda(x_2, y_{2n+1}), dy = x^{n+1}$.

(4) $\Lambda(T^n) = \Lambda(x_1^1, x_1^2, \cdots, x_1^n), d = 0$.

In the next setion we will describe the minimal model of a nilmanifold in terms of the structure of its defining nilpotent group.

In order to understand category in the framework of minimal models, assume for the moment that $\mathrm{cat}(M) = m$. The Whitehead diagram

$$(*) \qquad \begin{array}{ccc} M & \xrightarrow{\Delta} & M^{m+1} \\ & \Delta' \searrow & \uparrow j \\ & & T^{m+1}(M) \end{array}$$

translates (via Sullivan's categorical equivalence) into a homotopy commutative diagram of minimal cdga's,

$$(**) \qquad \begin{array}{ccc} \Lambda X & \leftarrow & (\Lambda X)^{\otimes m+1} \\ & \nwarrow \rho & \downarrow \xi \\ & & \Lambda Y \end{array}$$

where $\Lambda(M) = \Lambda X$, $\Lambda(M^{m+1}) = (\Lambda X)^{\otimes m+1}$ (since the model of a product is the tensor product of the models), Δ is modelled by the $(m+1)$-fold multiplication μ and $\Lambda Y = \Lambda(T^{m+1}(M))$.

Now, however, we may make the following

Definition. The *rational category* of M (or $\Lambda(M) = \Lambda X$), $\mathrm{cat}_0(M)$, is the least m so that (**) exists; that is, there exists ρ with $\rho\xi \simeq \mu$.

Observe that: (1) $\mathrm{cat}_0(M) \leqslant \mathrm{cat}(M)$ since any diagram (*) induces a diagram (**). (2) If M is simply connected, then $\mathrm{cat}_0(M) = \mathrm{cat}(M_0)$, where M_0 is the $\mathbf{Q}$-localization of M. This follows since (*) itself localizes.

The definition of $\mathrm{cat}_0(M)$ would be of little use if this were its only description. The passage from (*) to (**) simply transfers the difficult problem of obtaining Δ' to an (almost) equally difficult problem of obtaining ρ. However, by understanding the nature of $\Lambda Y = \Lambda(T^{m+1}(M))$, a more accessible criterion for $\mathrm{cat}_0(M)$ may be developed. We first describe ΛY.

PROPOSITION (2.2 of [3]). *A minimal model for the fat wedge is given by a minimal model* $\phi: \Lambda Y \to \Omega$ *for the quotient* cdga

$$\Omega = (\Lambda X)^{\otimes m+1}/(\Lambda^+ X)^{\otimes m+1}$$

where $\Lambda^+ X$ *consists of all elements of positive degree. Moreover, if* $\pi: (\Lambda X)^{\otimes m+1} \to \Omega$ *is the projection, then any* $\eta: (\Lambda X)^{\otimes m+1} \to \Lambda Y$ *with* $\phi\eta \simeq \pi$ *is homotopic to the induced map* ξ.

(The existence of η is a consequence of the minimality of $(\Lambda X)^{\otimes m+1}$, the fact that ϕ induces an isomorphism of cohomology and cdga obstruction theory. See [4] or [6].)

In some sense, the form of Ω is exactly what one would expect viewing the fat wedge as a spatial bound on the "form product" length (as opposed to cuplength). The proof of the proposition relies on various technical results involving $A(T^{m+1}(M))$.

Now let $\Lambda^{>m}X$ denote the differential ideal of ΛX having additive basis the monomials $x_{i_1} \cdots x_{i_k}$ with $k > m$. Consider the projection $p: \Lambda X \to \Lambda X/\Lambda^{>m}X$ and a minimal model $\theta: \Lambda Z \to \Lambda X/\Lambda^{>m}X$. As before (for ΛY), minimal model theory provides a lift of $p, \tilde{p}: \Lambda X \to \Lambda Z$, with $\theta\tilde{p} \simeq p$.

Say that ΛX is a *retract* of $\Lambda X/\Lambda^{>m}X$ if there exists a cdga map $r: \Lambda Z \to \Lambda X$ with $r\tilde{p} \simeq 1_{\Lambda X}$.

We are now in a position to give the rational homotopy criterion for category. We give a proof in one direction and refer to [3] for the other. (Also, we make use of the fact that a cohomology isomorphism $\theta: A \to B$ induces bijections of cdga homotopy sets $\theta_*: [\Lambda, A] \xrightarrow{\approx} [\Lambda, B]$ for any minimal Λ.) With the notation above, we have the

THEOREM. $\mathrm{cat}_0(M) \leqslant m$ *if and only if* $\Lambda X = \Lambda(M)$ *is a retract of* $\Lambda X/\Lambda^{>m}X$.

Proof. We only prove the "if" part. Let r denote the retraction, $\Lambda Z \to \Lambda X$, with $r\tilde{p} \simeq 1_{\Lambda X}$. We have the following homotopy commutative diagram (where $\bar{\mu}$ is the map induced by μ and $\tilde{\mu}$ is a lift to models),

$$\begin{array}{ccc}
\Lambda X & \overset{\mu}{\leftarrow} & (\Lambda X)^{\otimes m+1} \\
\downarrow p & & \downarrow \pi \\
\dfrac{\Lambda X}{\Lambda^{>m}X} & \overset{\bar\mu}{\leftarrow} & \dfrac{(\Lambda X)^{\otimes m+1}}{(\Lambda^{+}X)^{\otimes m+1}} \\
\simeq \uparrow \theta & & \simeq \uparrow \phi \\
\Lambda Z & \overset{\tilde\mu}{\leftarrow} & \Lambda Y
\end{array}
\qquad \tilde p: \Lambda X \to \Lambda Z, \quad \xi: (\Lambda X)^{\otimes m+1} \to \Lambda Y$$

In order to prove $\text{cat}_0(M) \leqslant m$, we must find $\rho: \Lambda Y \to \Lambda X$ with $\rho\xi \simeq \mu$. We can use the given retraction to do exactly this. Let $\rho = r\tilde\mu$.

First, observe $\theta\tilde p\mu \simeq p\mu = \bar\mu\pi \simeq \bar\mu\phi\xi \simeq \theta\tilde\mu\xi$. Because θ is a cohomology isomorphism, $\tilde p\mu \simeq \tilde\mu\xi$.

Now, $\rho\xi = r\tilde\mu\xi \simeq r\tilde p\mu \simeq 1_{\Lambda X}\mu = \mu$ and we are done. □

Of course, $\text{cat}_0(M)$ is, in general, too hard to compute. However, the criterion we have described opens up the possibility of defining weaker invariants which *are* computable. In a sense, the point of this paper is to give an exposition of these weaker invariants in the context of a specific problem of interest to "geometers".

Define $e_0(M)$ to be the least integer s so that $p: \Lambda X \to \Lambda X/\Lambda^{>s}X$ induces an injection in cohomology. (This is, in fact, equivalent to requiring $r: \Lambda Z \to \Lambda X$ to be only a *linear* retraction. The invariant $e_0(M)$ was first defined by Toomer [9] in terms of the Milnor-Moore spectral sequence.)

Note that if $r: \Lambda Z \to \Lambda X$ is a retraction, then $\tilde p^*$ is injective and (since θ^* is an isomorphism) therefore so is p^*. Hence, we clearly have

$$e_0(M) \leqslant \text{cat}_0(M)\ .$$

Moreover, when M is a nilpotent space (so that the full power of the minimal model may be utilized) *and* a manifold (so that Poincaré duality may be exploited), we can identify $e_0(M)$ in the following manner:

PROPOSITION. *If M^n is a nilpotent manifold with fundamental class $\tau \in H^n(M; \mathbf{Q})$, then $e_0(M)$ is the largest k such that τ is represented by a cocycle in $\Lambda^{\geqslant k}X$.*

Proof. Let $e_0(M) = s$ and let k be defined by the stated property. If τ is represented by a cocycle in $\Lambda^{>s}X$, then (for $p: \Lambda X \to \Lambda X/\Lambda^{>s}X$) $p^*(\tau) = 0$ and p^* is therefore not injective. Hence, $k \leqslant s$.

In order to show the reverse inequality $s \leqslant k$, we must show that, for $p: \Lambda X \to \Lambda X/\Lambda^{>k}X, p^*$ is injective. Plainly, by Poincaré duality, p^* is injective if and only if $p^*(\tau) \neq 0$. Hence, we prove this.

Suppose $p^*(\tau) = 0$. Let τ denote the representing cocycle in $\Lambda^{\geqslant k}X$ of the fundamental class τ. Let $p(\tau) = \bar{\tau} \in \Lambda X/\Lambda^{>k}X$ and consider $\bar{\tau}$ as an element in $\Lambda^{\leqslant k}X$. Now, $p^*(\tau) = 0$, so there exists $\bar{\alpha} \in \Lambda X/\Lambda^{>k}X$ with $\bar{d}\bar{\alpha} = \bar{\tau}$. Consider $\bar{\alpha} \in \Lambda^{\leqslant k}X$ as well and note that $p(d\bar{\alpha}) = \bar{d}\bar{\alpha} = \bar{\tau}$. Therefore, in ΛX

$$d\bar{\alpha} = \bar{\tau} + \Phi\,, \qquad \text{where } \Phi \in \Lambda^{>k}X\,.$$

Similarly, of course, $\tau = \bar{\tau} + \Omega$ for $\Omega \in \Lambda^{>k}X$ and we obtain,

$$\tau = \bar{\tau} + \Omega = d\bar{\alpha} - \Phi + \Omega$$

with $\Omega - \Phi \in \Lambda^{>k}X$. But this means τ is cohomologous to $\Omega - \Phi \in \Lambda^{>k}X$, contradicting the definition of k. □

§3. NILMANIFOLDS

A *nilmanifold* M is the quotient of a nilpotent Lie group N by a discrete cocompact subgroup π. The description below follows [7].

It is well known that N is diffeomorphic to some $\mathbf{R}^n$ and, therefore, M is a $K(\pi, 1)$. Furthermore, this entails the fact that π is a finitely generated torsionfree nilpotent group.

On the algebraic side, there is a refinement of the upper central series of π,

$$\pi \supseteq \pi_2 \supseteq \pi_3 \supseteq \cdots \supseteq \pi_n \supseteq 1$$

with each $\pi_i/\pi_{i+1} \cong \mathbf{Z}$ whose length is invariant and is called the *rank* of π. So, for π above, $\operatorname{rank}(\pi) = n$.

This description implies that any $u \in \pi$ has a decomposition $u = u_1^{x_1} \cdots u_n^{x_n}$, where $\langle u_n \rangle = \pi_n, \cdots \langle u_i \rangle = \pi_i/\pi_{i+1}$. The set $\{u_1, \cdots u_n\}$ is called a Malcev basis for π. Using this basis the multiplication in π takes the form

$$u_1^{x_1} \cdots u_n^{x_n} u_1^{y_1} \cdots u_n^{y_n} = u_1^{\rho_1(x,y)} \cdots u_n^{\rho_n(x,y)}$$

where

$$\rho_i(x, y) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})\,.$$

Example. $N = U_n(\mathbf{R})$, the group of upper diagonal matrices with 1's on the diagonal; $\pi = U_n(\mathbf{Z})$. A Malcev basis is given by $\{u_{ij} \mid 1 \leqslant i < j \leqslant n\}$ where $u_{ij} = I + e_{ij}$ and

$$\rho_{ij}(x, y) = x_{ij} + y_{ij} + \sum_{i<k<j} x_{ik} y_{kj} .$$

Consider the central extension $\pi_n \to \pi \to \bar{\pi}$. The cocycle for the extension is $\tau_n \colon \bar{\pi} \times \bar{\pi} \to \mathbf{Z}$. Of course $\bar{\pi}$ is also finitely generated torsionfree with refined upper central series,

$$\bar{\pi} = \frac{\pi}{\pi_n} \supseteq \frac{\pi_2}{\pi_n} \supseteq \cdots \supseteq \frac{\pi_{n-1}}{\pi_n} \supseteq \frac{\pi_n}{\pi_n} = 1 .$$

Hence, $\operatorname{rank}(\bar{\pi}) = n - 1$ and

$$\bar{\rho}_i(x, y) = \rho_i((x, 0), (y, 0)) = x_i + y_i + \tau_i(x_1, \cdots x_{i-1}, y_1, \cdots y_{i-1})$$

for $i < n$. Clearly, then, we may iterate this process and decompose π as n central extensions of the form

$$\mathbf{Z} \to G \to \bar{G}$$

with cocycles $\tau_i \in H^2(\bar{G}; \mathbf{Z})$ (with untwisted coefficients since the extension is central).

This desription allows a geometric formulation:

$$\tau_n \in H^2(\bar{\pi}; \mathbf{Z}) \cong H^2(K(\bar{\pi}, 1); \mathbf{Z}) \cong [K(\bar{\pi}, 1), K(\mathbf{Z}, 2)]$$

by the usual identification of cohomology groups with sets of homotopy classes into $K(\mathbf{Z}, m)$'s. Now, $K(\mathbf{Z}, 2) = \mathbf{C}P(\infty)$, the classifying space for principal S^1-bundles, so τ_n induces a bundle over $K(\bar{\pi}, 1)$,

$$\begin{array}{ccc} S^1 & \to & K(\pi, 1) \\ & & \downarrow \\ & & K(\bar{\pi}, 1) \overset{\tau_n}{\to} \mathbf{C}P(\infty) . \end{array}$$

The total space of the bundle is clearly $K(\pi, 1)$ since the ensuing short exact sequence of fundamental groups is classified by τ_n.

Now, because we can iterate the algebraic decomposition of π, we obtain an iterated sequence of principal S^1-bundles classifed by the τ_i:

$$\begin{array}{ccccc}
S^1 & \to & M & = & K(\pi, 1) \\
 & & \downarrow & & \\
S^1 & \to & M_{n-1} & \xrightarrow{\tau_n} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
S^1 & \to & M_1 & \xrightarrow{\tau_2} & \mathbf{C}P(\infty) \\
 & & \downarrow & & \\
 & & * & \xrightarrow{\tau_1} & \mathbf{C}P(\infty) .
\end{array}$$

We can assume (by finite dimensionality) that each τ_i has image in a finite $\mathbf{C}P(n)$, so thus may be approximated by a smooth map. Hence, each M_j is a compact manifold with

$$\dim(M_j) = \dim(M_{j-1}) + 1 .$$

Thus, $\dim(M) = \operatorname{rank}(\pi) = n$.

§4. Category of nilmanifolds

The decomposition of $M = K(\pi, 1)$ into a tower of principal S^1-bundles is, in fact, the Postnikov decomposition of M with k-invariants the τ_i. By the fundamental theorem of rational homotopy theory, the minimal model has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_n), d) , \quad \deg(x_i) = 1$$

with $dx_i = \tau_i$, where τ_i is a cocycle representing the class $\tau_i \in H^2(M_{i-1}; \mathbf{Z})$. Note that $\Lambda(M)$ is an exterior algebra because all generators are in degree 1. Therefore, since $\dim M = n$, the only possibility for a cocycle representing the fundamental class is $x_1 \cdots x_n$. Hence, $e_0(M) = n$ and this immediately implies,

Proof of Theorem 1. $n = e_0(M) \leqslant \operatorname{cat}_0(M) \leqslant \operatorname{cat}(M) \leqslant \dim M = n$. □

Example. Consider the 3-dimensional Heisenberg group $U_3(\mathbf{R})$ and mod out by $U_3(\mathbf{Z})$. The resulting M is a 3-manifold obtained as a principal bundle,

$$S^1 \to M \to T^2$$

with classifying element (over the rationals) $xy \in H^2(T^2; \mathbf{Q})$, where x and y are one-dimensional generators. The minimal model of M is then given by

$$\Lambda(M) = \Lambda(x, y, z) \qquad \deg(x) = \deg(y) = \deg(z) = 1$$

with $dx = 0 = dy$ and $dz = xy$. Additive generators for cohomology are then,

$H^1 : x, y$

$H^2 : xz, yz$ (Massey products!)

$H^3 : zyx$.

Note that $\text{cup}(M) = 2$, but $\text{cat}(M) = 3$.

In some sense then, the proof of Theorem 1 is simply an observation that the techniques of rational homotopy theory work particularly well for nilmanifolds.

PROBLEM. If π is not nilpotent, then a $K(\pi, 1)$ is not a nilpotent space, so the minimal model does not describe a "rational type". Is it possible, however, that enough information about a $K(\pi, 1)$ is present in the model to determine its category (in the compact case say)?

§5. HIGHER DEGREE ANALOGUES

An analogue of the minimal model of a nilmanifold is one of the form,

$$(\Lambda(x_1, \cdots x_n), d) , \qquad \text{degree}(x_i) = \text{odd} .$$

Such an algebra is known to satisfy rational Poincaré duality (see [5]) and to have formal top dimension $\sum_i \deg(x_i)$. But, plainly, the same argument as before applies to show that the "only" element in this exterior algebra which can reach the stated dimension is $x_1 \cdots x_n$. Hence (since this is the longest product in Λ), the fundamental class is maximally represented by a product of length n and

LEMMA. $e_0(\Lambda) = n$.

Now, we may consider Λ as built up by adjoining odd generators one at a time (with decomposable differential). Let ΛZ be a minimal cdga and y of odd degree. Then

PROPOSITION. (See Theorem 4.7 and Lemma 6.6 of [3].)

$$\text{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant \text{cat}_0(\Lambda Z) + 1 .$$

Proof. Suppose $\mathrm{cat}_0(\Lambda Z) = m$. Then ΛZ is a retract of $\Lambda Z/\Lambda^{>m}Z$ and we see that $\Lambda Z \otimes \Lambda y$ is a retract of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$. Now, the maximal product length of $\Lambda Z/\Lambda^{>m}Z \otimes \Lambda y$ is $m + 1$ and this is sufficient to ensure $\mathrm{cat}_0(\Lambda Z \otimes \Lambda y) \leqslant m + 1$. □

Now, by induction, we see that $\mathrm{cat}_0(\Lambda) \leqslant n$ (since for x_1 of odd degree $\mathrm{cat}_0(\Lambda x_1) = 1$). Putting this together with the Lemma gives

THEOREM 2. *If* $\Lambda = (\Lambda(x_1, \cdots x_n), d)$ *with* $\deg(x_i) = \text{odd}$ *for each* i, *then* $\mathrm{cat}_0(\Lambda) = n$.

This result may be applied, for example, to a manifold obtained as an iterated principal bundle. That is, for compact Lie groups G_i, $i = 1$ to N.

$M_1 = G_1$; M_i is obtained from M_{i-1} as a principal G_i-bundle over M_{i-1}.

$M = M_N$

Each G_i is, rationally, a product of $\mathrm{rank}(G_i)$ odd spheres, so the minimal model of M has the form,

$$\Lambda(M) = (\Lambda(x_1, \cdots x_s), d)$$

with $\deg(x_i) = \text{odd}$ and $s = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. $\mathrm{cat}_0(M) = \sum_{i=1}^{N} \mathrm{rank}(G_i)$.

COROLLARY. *If* M *is an iterated principal bundle with fibres* G_i, *then the number of critical points of any smooth function on* M *is bounded below by* $\sum_i \mathrm{rank}(G_i) + 1$.

Note that we have not determined $\mathrm{cat}(M)$, so the true effectiveness of Lusternik-Schnirelmann theory may not have been exploited.

§6. GANEA'S CONJECTURE

The Ganea Conjecture states that, for a finite CW complex X, $\mathrm{cat}(X \times S^k) = \mathrm{cat}(X) + 1$ for any sphere S^k. Although unproven in general, various cases of the conjecture have been shown to be true. We add nilmanifolds to that list:

THEOREM. *Ganea's Conjecture is true for nilmanifolds.*

Proof. Let M be a nilmanifold. Then

$$\begin{aligned}\dim M + 1 &= e_0(M) + 1 \\ &= e_0(M \times S^k) \text{ since } e_0 \text{ respects products} \\ &\leqslant \operatorname{cat}(M \times S^k) \\ &\leqslant \operatorname{cat}(M) + 1 \text{ Fox's inequality} \\ &= \dim M + 1 .\end{aligned}$$

Hence all inequalities are equalities and $\operatorname{cat}(M \times S^k) = \operatorname{cat}(M) + 1$. □

ADDED IN PROOF. By using the equality $e_0(M) = \dim(M)$ and extending the e_0-invariant to maps, C. McCord and the author have given a proof of the Arnold Conjecture for nilmanifolds (cf. C. McCord and J. Opera, *Rational Ljusternik-Schnirelmann Category and the Arnold Conjecture for Nilmanifolds,* preprint 1992). That is, any smooth 1-periodic Hamiltonian system on a symplectic nilmanifold M has at least $\dim(M) + 1$ contractible 1-periodic orbits.

REFERENCES

[1] ABRAHAM, R. and J. R. MARSDEN. *Foundations of Mechanics, 2nd Ed.* (1985 version) Addison-Wesley, 1978.

[2] FELIX, Y. La Dichotomie Elliptique-Hyperbolique. In *Homotopie Rationnelle, Astérisque 176,* Soc. Math. France, 1989.

[3] FELIX, Y. and S. HALPERIN. Rational L. S. Category and Its Applications. *Trans. Amer. Math. Soc. 273* (1982), 1-73.

[4] GRIFFITHS, P. and J. MORGAN. *Rational Homotopy Theory and Differential Forms.* Progress in Math. 16, Birkhaüser, 1981.

[5] HALPERIN, S. Finiteness in the Minimal Models of Sullivan. *Trans. Amer. Math. Soc. 230* (1977), 173-199.

[6] —— *Lectures on Minimal Models.* Mémoires Soc. Math. France 9-10 (1983).

[7] LAMBE, L. and S. PRIDDY. Cohomology of Nilmanifolds and Torsionfree Nilpotent Groups. *Trans. Amer. Math. Soc. 273* (1982), 39-55.

[8] LUSTERNIK, L. and L. SCHNIRELMANN. *Méthodes topologiques dans les problèmes variationnels.* Actualités Scientifiques et Industrielles 188, Hermann et Cie, Paris, 1934.

[9] TOOMER, G. H. L. S. Category and the Moore Spectral Sequence. *Math. Z. 138* (1974), 123-143.

[10] WHITEHEAD, G. *Elements of Homotopy Theory*. Grad. Texts in Math. 61, Springer-Verlag, 1978.
[11] EILENBERG, S. and T. GANEA. On the Lusternik-Schnirelmann category of abstract groups. *Annals of Math. 65, no. 3* (1957), 517-518.
[12] HILTON, P., G. MISLIN and J. ROITBERG. *Localization of Nilpotent Groups and Spaces*. North Holland, Amsterdam, 1975.

(*Reçu le 8 janvier 1991*)

John Oprea
Department of Mathematics
Cleveland State University
Euclid Avenue at East 24th Street
Cleveland, Ohio 44115 (USA)