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THE CATEGORY OF NILMANIFOLDS

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ABSTRACT. The techniques of rational homotopy theory are used to compute the category of a nilmanifold: $\operatorname{cat}(M) = \dim M = \operatorname{rank}(\pi_1 M)$. This information is of interest to dynamicists since the theorem of Lusternik-Schnirelmann then shows that the number of critical points of a smooth function of M is bounded below by $\operatorname{rank}(\pi_1 M) + 1$.

INTRODUCTION

As a first step to understanding the structure of certain dynamical systems on nilmanifolds, one might hope to have computable lower bounds on the number of critical points of smooth functions. Of course, one is then led to the Lusternik-Schnirelmann definition of category and their well-known result that category (+ 1) is such a bound. Unfortunately, category is rarely computable, so those who require numerical bounds often employ the fact that category majorizes cuplength. Hence cuplength (which, generally, is a more computable homotopy invariant than category) is the numerical invariant frequently sought for in order to provide a lower bound for the number of critical points of smooth functions on a manifold.

Indeed, some time ago, for the reasons above, Chris McCord asked me if I knew of a formula for the cuplength of a nilmanifold. I did not then, and after many computations I do not now! Thus, I pose:

QUESTION. What is the cuplength (with $\mathbf{Q}$-coefficients say) of a nilmanifold?

Suprisingly, however, the need for such knowledge by dynamicists is obviated by the following.